Minimal Surfaces and H-Surfaces

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Abstract

This thesis illustrates the existence of minimal surfaces bounded by a prescribed closed Jordan curve $\Gamma \subset \mathbb{R}^3$. Furthermore, it discusses the existence of surfaces of non-vanishing constant mean curvature $H \in \mathbb{R}$ bounded by Γ. In the final chapter it concludes by investigating the boundary behaviour of minimal surfaces and H-surfaces. It simplifies the discussion of Erhard Heinz and Friedrich Tomi on the boundary regularity of minimal surfaces and extends their results onto H-surfaces.

Contents

1 Introduction **1**

1.1 Differential Geometrical Consideration 1

1.2 Euler–Lagrange Equations 5

1.3 Generalised Minimal Surfaces 6

2 Two Boundary Value Problems **9**

2.1 Minimal Surfaces 9

2.1.1 The Classical Plateau Problem 10

2.1.2 Variational Formulation 13

2.1.3 Calculus of Variations 14

2.1.4 Three-Point Condition 16

2.1.5 Non-emptiness 16

2.1.6 Courant–Lebesgue Lemma 18

2.1.7 Existence 20

2.1.8 Harmonicity 23

2.1.9 Conformality 23

2.1.10 Plateau's Boundary Condition 27

2.1.11 Solution of Plateau's Problem 28

2.1.12 A-priori Bound 29

2.2 H-Surfaces 29

2.2.1 The Volume Functional 32

2.2.2 Existence 33

2.2.3 Continuity 35

3 Boundary Regularity **36**
3.1 Heinz and Tomi on the Boundary Behaviour of Minimal Surfaces . 38
3.1.1 A Regularity Result 39
3.1.2 First Main Result 44
3.1.3 Second Main Result 48
3.2 Boundary Regularity for Minimal Surfaces 58
3.3 Boundary Regularity for H-Surfaces 59

Bibliography **61**

Chapter 1

Introduction

1.1 Differential Geometrical Consideration

In this section we introduce minimal surfaces by closely following the geometrical considerations of Stefan Hildebrandt in [13] and Manfredo Perdigao do Carmo [2]. The aim is to characterise minimal surfaces as critical points of the area functional.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain in $\mathbb{R}^2$. Let us consider a surface $X : \bar{\Omega} \to \mathbb{R}^3$ as a C^2-immersion of $\bar{\Omega}$ into $\mathbb{R}^3$. For vectors $a, b \in \mathbb{R}^3$ we denote the scalar product in the Euclidean space by $a \cdot b$ and the length by $|a| = \sqrt{a \cdot a}$. More generally, we denote the inner product of a and b by $\langle a, b \rangle$. We write X in the form

$$X(u, v) = \big(X^1(u, v), X^2(u, v), X^3(u, v)\big),$$

where $w = (u, v) \in \bar{\Omega} \subset \mathbb{R}^2$. We denote the partial derivatives of X with respect to u and v as X_u and X_v, respectively. For the gradient of X we write $\nabla X = \big(X_u, X_v\big)$ with $|\nabla X| = |X_u|^2 + |X_v|^2$.

We define the first fundamental form I_p on the tangent plane $T_p(X)$ of the regular surface $X \subset \mathbb{R}^3$ at point $p \in X$ by

$$I_p(w) = \langle w, w \rangle_p = |w|^2 \geq 0,$$

where $\langle w_1, w_2 \rangle_p$ denotes the inner product of $w_1, w_2 \in T_p(X)$ viewed as vectors in $\mathbb{R}^3$. Consider a parametrised curve $\alpha(t) = X(u(t), v(t)),\ t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Then we can express the first fundamental form in the basis $\{X_u, X_v\}$ of $T_p(X)$ with $p = \alpha(0)$ as

$$\begin{aligned}
I_p(\alpha'(0)) &= \langle \alpha'(0), \alpha'(0) \rangle_p \\
&= \langle X_u u' + X_v v', X_u u' + X_v v' \rangle_p \\
&= \langle X_u, X_u \rangle_p \big(u'\big)^2 + 2\langle X_u, X_v \rangle_p u'v' + \langle X_v, X_v \rangle_p \big(v'\big)^2 \\
&= \mathcal{E}\big(u'\big)^2 + 2\mathcal{F}u'v' + \mathcal{G}\big(v'\big)^2,
\end{aligned}$$

where the coefficients of the first fundamental form are given by

$$\begin{aligned}
\mathcal{E}(u(0), v(0)) &= \langle X_u, X_u \rangle_p, \\
\mathcal{F}(u(0), v(0)) &= \langle X_u, X_v \rangle_p, \\
\mathcal{G}(u(0), v(0)) &= \langle X_v, X_v \rangle_p.
\end{aligned}$$

Letting p run through a coordinate neighbourhood of $X(u, v)$ we obtain functions $\mathcal{E}(u, v)$, $\mathcal{F}(u, v)$, $\mathcal{G}(u, v)$ and we write

$$\begin{aligned}
\mathcal{E} &= \langle X_u, X_u \rangle = |X_u|^2, \\
\mathcal{F} &= \langle X_u, X_v \rangle = |X_u \cdot X_v|, \\
\mathcal{G} &= \langle X_v, X_v \rangle = |X_v|^2
\end{aligned} \tag{1.1}$$

for the coefficients of the first fundamental form.

The area element dA of X is given by $dA = |X_u \wedge X_v|\, du\, dv$, where $X_u \wedge X_v$ denotes the exterior product of X_u and X_v. Therefore the area functional reads

$$A(X) = \int_\Omega dA = \int_\Omega |X_u \wedge X_v|\, du\, dv. \tag{1.2}$$

Observing that

$$|X_u \wedge X_v|^2 + \langle X_u, X_v \rangle^2 = |X_u|^2 |X_v|^2$$

we write

$$|X_u \wedge X_v| = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2} =: \mathcal{W}.$$

Let $S^2 := \{a \in \mathbb{R}^3 \mid |a| = 1\}$ be the 2-sphere. The surface normal of X is the map

$N : \bar{\Omega} \to S^2$ given by

$$N = \frac{1}{\mathcal{W}} X_u \wedge X_v.$$

For the parametrised curve $\alpha(t) = X(u(t), v(t)), t \in (-\varepsilon, \varepsilon)$ on X with $\alpha(0) = p$, the tangent vector to $\alpha(t)$ at p is $\alpha'(0) = X_u u' + X_v v'$. Moreover, there holds

$$dN(\alpha'(0)) = N_u u' + N_v v'.$$

Now we can express the second fundamental form of X in the basis $\{X_u, X_v\}$ as

$$\begin{aligned} II_p(\alpha'(0)) &= -\langle dN(\alpha'(0)), \alpha'(0)\rangle \\ &= -\langle N_u u' + N_v v', X_u u' + X_v v'\rangle \\ &= \mathcal{L}(u')^2 + 2\mathcal{M}u'v' + \mathcal{N}(v')^2, \end{aligned}$$

where

$$\begin{aligned} \mathcal{L} &= -\langle N_u, X_u\rangle = \langle N, X_{uu}\rangle, \\ \mathcal{M} &= -\langle N_v, X_u\rangle = \langle N, X_{uv}\rangle = \langle N, X_{vu}\rangle = -\langle N_u, X_v, \rangle \\ \mathcal{N} &= -\langle N_v, X_v\rangle = \langle N, X_{vv}\rangle, \end{aligned} \tag{1.3}$$

since $\langle N, X_u\rangle = \langle N, X_v\rangle = 0$, confer [2].

Then we can express the mean curvature H of X as

$$H = \frac{1}{2}\frac{\mathcal{L}\mathcal{G} - 2\mathcal{M}\mathcal{F} + \mathcal{N}\mathcal{E}}{\mathcal{E}\mathcal{G} - \mathcal{F}^2}. \tag{1.4}$$

We now introduce the notion of a variation. Let $\varepsilon_0 > 0$. We choose a differentiable function $h : \bar{\Omega} \to \mathbb{R}$. The normal variation of $X(\bar{\Omega})$ determined by h is given by $Z : \bar{\Omega} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^3$ with

$$Z(u, v, \varepsilon) = X(u, v, \varepsilon) + \varepsilon h(u, v) N(u, v). \tag{1.5}$$

Fix $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Then the map $X^\varepsilon : \Omega \to \mathbb{R}^3$,

$$X^\varepsilon(u, v) = Z(u, v, \varepsilon)$$

is a parametrised surface with

$$
\begin{aligned}
X_u^\varepsilon &= X_u + \varepsilon h N_u + \varepsilon h_u N, \\
X_v^\varepsilon &= X_v + \varepsilon h N_v + \varepsilon h_v N.
\end{aligned}
$$

For the coefficients of the first fundamental form (1.1) we obtain

$$
\begin{aligned}
\mathcal{E}^\varepsilon &= \mathcal{E} + 2\varepsilon h\langle X_u, N_u\rangle + \varepsilon^2 h^2\langle N_u, N_u\rangle + \varepsilon^2 h_u h_u, \\
\mathcal{F}^\varepsilon &= \mathcal{F} + \varepsilon h\big(\langle X_u, N_v\rangle + \langle X_v, N_u\rangle\big) + \varepsilon^2 h^2\langle N_u, N_v\rangle + \varepsilon^2 h_u h_v, \\
\mathcal{G}^\varepsilon &= \mathcal{G} + 2\varepsilon h\langle X_v, N_v\rangle + \varepsilon^2 h^2\langle N_v, N_v\rangle + \varepsilon^2 h_v h_v.
\end{aligned}
$$

Thus we obtain with (1.3) and (1.4)

$$
\begin{aligned}
\mathcal{E}^\varepsilon\mathcal{G}^\varepsilon - \big(\mathcal{F}^\varepsilon\big)^2 &= \mathcal{E}\mathcal{G} - \mathcal{F}^2 - 2\varepsilon h\big(\mathcal{E}\mathcal{N} - 2\mathcal{F}\mathcal{M} + \mathcal{G}\mathcal{L}\big) + R \\
&= \big(\mathcal{E}\mathcal{G} - \mathcal{F}^2\big)\big(1 - 4\varepsilon h H\big) + R,
\end{aligned}
$$

where $\lim_{\varepsilon\to 0}\frac{R}{\varepsilon} = 0$. Hence if $\varepsilon_0 > 0$ is sufficiently small, X^ε is a regular parametrised surface with area

$$
\begin{aligned}
A(X^\varepsilon) &= \int_{\bar\Omega}\sqrt{\mathcal{E}^\varepsilon\mathcal{G}^\varepsilon - \big(\mathcal{F}^\varepsilon\big)^2}\,du\,dv \\
&= \int_{\bar\Omega}\sqrt{\big(\mathcal{E}\mathcal{G} - \mathcal{F}^2\big)}\sqrt{1 - 4\varepsilon h H + \frac{R}{\big(\mathcal{E}\mathcal{G} - \mathcal{F}^2\big)}}\,du\,dv.
\end{aligned}
$$

For small ε_0, the functional A is differentiable in ε, and we compute

$$
\frac{d}{d\varepsilon}A(X^\varepsilon)\Big|_{\varepsilon=0} = -\int_{\bar\Omega} 2hH\sqrt{\big(\mathcal{E}\mathcal{G} - \mathcal{F}^2\big)}\,du\,dv. \tag{1.6}
$$

This is the so called first variation of the area functional (1.2).

We are now led to characterise minimal surfaces properly following [2]. The observations above imply

Proposition 1.1.1. *Let $X : \bar\Omega \to \mathbb{R}^3$ be a regular parametrised surface. Then the first variation* (1.6) *of the area functional* (1.2) *vanishes for all normal variations* (1.5) *of X if and only if the mean curvature H* (1.4) *of X satisfies $H = 0$.*

Proof. If $H \equiv 0$, then (1.6) implies $\frac{d}{d\varepsilon}A(\varepsilon)\Big|_{\varepsilon=0} = 0$.

Conversely, assume that the first variation of A vanishes, i.e. $\frac{d}{d\varepsilon}A(\varepsilon)\Big|_{\varepsilon=0} = 0$. Suppose by contradiction $H(q) \neq 0$ for some $q \in \Omega$. Choose $h : \bar{\Omega} \to \mathbb{R}$ such that $h(q) = H(q)$, and h vanishes outside a small neighbourhood of q. Then (1.6) implies $\frac{d}{d\varepsilon}A(\varepsilon)\Big|_{\varepsilon=0} < 0$ for the normal variation determined by this h, contradicting the assumption. □

In other words, for a surface X of vanishing mean curvature there holds that $X(\bar{\Omega})$ is a critical point of the area functional A for any normal variation of $X(\bar{\Omega})$. This motivates the following definition [13].

Definition 1.1.1 (Minimal Surface)**.** A C^2-immersion $X : \Omega \to \mathbb{R}^3$ of a parameter domain $\Omega \subset \mathbb{R}^2$ is a *minimal surface* if its mean curvature H (1.4) satisfies $H = 0$.

1.2 Euler–Lagrange Equations

In this section, we derive the Euler–Lagrange equation for the area functional (1.2) introduced above. To this end, let $\Omega \subset \mathbb{R}^2$ be a domain in $\mathbb{R}^2$. Let us consider a surface $X : \bar{\Omega} \to \mathbb{R}^3$ such that X is described as a graph of a function $w : \bar{\Omega} \to \mathbb{R}$ of class C^2, that is

$$X(u,v) = (u, v, w(u,v)), \quad (u,v) \in \Omega.$$

Then

$$|X_u \wedge X_v| = \sqrt{1 + |\nabla w|^2},$$

and therefore the area functional (1.2) reads

$$A(X) = \int_\Omega dA = \int_\Omega |X_u \wedge X_v| \, du \, dv = \int_\Omega \sqrt{1 + |\nabla w|^2} \, dw =: \mathcal{A}(w).$$

We will now derive the minimal surface equation [27], by looking for a surface minimising the area functional. For this purpose let $\varphi \in C_c^\infty(\Omega)$. We compute

$$
\begin{aligned}
0 &= \frac{d}{d\varepsilon}\mathcal{A}(w+\varepsilon\varphi)\Big|_{\varepsilon=0} \\
&= \int_\Omega \frac{\nabla w + \varepsilon\nabla\varphi}{\sqrt{1+|\nabla w+\varepsilon\nabla\varphi|^2}}\nabla\varphi\, dw\Big|_{\varepsilon=0} \\
&= \int_\Omega \frac{\nabla w\nabla\varphi}{\sqrt{1+|\nabla w|^2}}\, dw \\
&= -\int_\Omega \nabla\cdot\big(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\big)\varphi\, dw
\end{aligned}
$$

where we used integration by parts in the last step. Therefrom we infer that w is stationary for $\mathcal{A}$ if and only if

$$\nabla\cdot\big(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\big) = 0 \quad \text{in } \Omega. \tag{1.7}$$

Equation (1.7) is the Euler–Lagrange equation for the non-parametric area functional $\mathcal{A}$. Equivalently,

$$w_{uu} + w_{vv} - \frac{w_{uu}w_u^2 + 2w_{uv}w_uw_v + w_{vv}w_v^2}{1+w_u^2+w_v^2} = 0,$$

or

$$(1+w_v^2)w_{uu} - 2w_{uv}w_uw_v + (1+w_u^2)w_{vv} = 0. \tag{1.8}$$

Equation (1.8) is the minimal surface equation. The surface X is minimal, i.e. is a critical point of $A(X)$ if and only if w is a critical point of $\mathcal{A}(w)$, i.e. if and only if w solves (1.8), see [13].

1.3 Generalised Minimal Surfaces

With Hildebrandt's reasoning in [13], we arrive at the following theorem.

Theorem 1.3.1. *Any C^2-immersion $X : \Omega \to \mathbb{R}^3$ of vanishing mean curvature is equivalent to an immersed surface represented by conformal parameters, i.e. there exists a diffeomorphism $\varphi : \Omega \to \Omega^*$ onto some simply connected domain Ω^* with inverse $\psi : \Omega^* \to \Omega$ such that $Y(u,v) := X(\psi(u,v))$ satisfies the conformality* *relations*

$$|Y_u|^2 = |Y_v|^2, \quad Y_u \cdot Y_v = 0. \tag{1.9}$$

We will not discuss the proof here. The theorem shall only elucidate that it suffices to consider surfaces $X : \Omega \to \mathbb{R}^3$ of class C^2 satisfying

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0. \tag{1.10}$$

For these surfaces we find

Theorem 1.3.2. *An immersion* $X \in C^2(\Omega, \mathbb{R}^3)$ *with* (1.10) *satisfies*

$$\Delta X = 2H X_u \wedge X_v,$$

where $\Delta X = X_{uu} + X_{vv}$ *denotes the Laplacian.*

Proof. Note that (1.1) implies that (1.10) is equivalent to $\mathcal{E} = \mathcal{G}$ and $\mathcal{F} = 0$. This means for the mean curvature (1.4)

$$H = \frac{\mathcal{L} + \mathcal{N}}{2\mathcal{E}} = \frac{\langle X_{uu}, N\rangle + \langle X_{vv}, N\rangle}{2\mathcal{E}} = \frac{\langle \Delta X, N\rangle}{2\mathcal{E}}. \tag{1.11}$$

On the other hand, by differentiating (1.10) in u and v we obtain

$$\begin{aligned}
\langle X_u, X_{uu}\rangle &= \langle X_v, X_{vu}\rangle,\\
\langle X_u, X_{uv}\rangle &= \langle X_v, X_{vv}\rangle,\\
\langle X_{uv}, X_v\rangle + \langle X_u, X_{vv}\rangle &= 0,\\
\langle X_{uu}, X_v\rangle + \langle X_u, X_{vu}\rangle &= 0.
\end{aligned}$$

Thus by symmetry of the scalar product and using Schwarz's theorem to exchange the order of the partial derivatives we find

$$\begin{aligned}
\langle X_u, \Delta X\rangle &= 0,\\
\langle X_v, \Delta X\rangle &= 0,
\end{aligned}$$

i.e. ΔX is perpendicular to both X_u and X_v. Furthermore, (1.10) gives

$$|X_u \wedge X_v| = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2} = \mathcal{E}.$$

Thus $X_u \wedge X_v = \langle \mathcal{E}, N \rangle$. Combining this with (1.11) we conclude

$$\Delta X = 2H\langle \mathcal{E}, N \rangle = 2HX_u \wedge X_v.$$

$\square$

Theorem 1.3.2 directly implies

Corollary 1.3.2.1. *A conformal C^2-immersion $X : \Omega \to \mathbb{R}^3$ is a minimal surface if and only if X is harmonic.*

This corollary concludes the introductory section.

Chapter 2

Two Boundary Value Problems

2.1 Minimal Surfaces

Minimal surfaces arise as solutions to Plateau's problem, named after the Belgian mathematician Joseph Plateau, whose experiments with soap in 1873 would entail a challenge for mathematicians in the subsequent century to determine a mathematical formulation to describe them. By forming a frame of iron wire and dipping it into soap liquid Plateau formed a film, the figure of which was the surface of least area which has the frame for its boundary [18]. Mathematically, in Plateau's experiment, the wire corresponds to a closed Jordan curve of finite length, and the soap film to a two-dimensional surface in the Euclidean space $\mathbb{R}^3$. The soap film in stable equilibrium corresponds to a surface of least area. Plateau's problem consists of finding a disc-type minimal surface of least area spanning a closed simple curve.

Note that a regular surface of least area has vanishing mean curvature, hence it is a minimal surface. Instead of finding an absolute minimiser of the area, we can more generally look for minimal surfaces spanning a closed simple curve. The latter corresponds to determining stationary points of the area functional [13], the existence of which was first proved by Jesse Douglas [4] and Tibor Radò [24] in 1930. In this section we will establish the existence of solutions to Plateau's problem following the approach of minimising area amongst surfaces given as maps from a two-dimensional parameter domain into $\mathbb{R}^3$. We follow Richard Courant's [3] simplification of Douglas's proof.

2.1.1 The Classical Plateau Problem

We will restrict our considerations to surfaces $X \in C^0(\bar{B};\mathbb{R}^3)$ parametrised on the closure of the unit disc

$$B = \{w = (u,v) \in \mathbb{R}^2 \big| \ u^2 + v^2 < 1\}.$$

The space $C^0(\bar{B};\mathbb{R}^3)$ denotes the space of continuous functions mapping the domain $\bar{B}$ into $\mathbb{R}^3$. For later reference, we introduce the spaces $C^m(\Omega;\mathbb{R}^n)$ of m-times continuously differentiable functions from some open domain $\Omega \subset \mathbb{R}^d$ into $\mathbb{R}^n$ for some $d, n \in \mathbb{N}$. We set $C^\infty(\Omega;\mathbb{R}^n) := \bigcap_{k=1}^{\infty} C^k(\Omega;\mathbb{R}^n)$ and the space of vectors $z : \Omega \to \mathbb{R}^n$ of class C^∞ with compact support is denoted by $C_c^\infty(\Omega;\mathbb{R}^n)$. We equip these spaces with the usual supremum norm

$$\begin{aligned} \|z\|_{C^0(\Omega;\mathbb{R}^n)} &:= \sup_{w\in\Omega} |z(w)|, \\ \|z\|_{C^m(\Omega;\mathbb{R}^n)} &:= \sum_{|\alpha|\leq m} \|D^\alpha z\|_{C^0(\Omega;\mathbb{R}^n)} \end{aligned}$$

for $z : \Omega \to \mathbb{R}^n$. Moreover, we denote by $H^1(\Omega;\mathbb{R}^n)$ the Sobolev space of square-integrable functions $z : \Omega \to \mathbb{R}^n$, whose first distributional derivatives are again in $L^2(\Omega;\mathbb{R}^n)$. We denote the L^2-norm, the H^1-seminorm and the H^1-norm on Ω by

$$\begin{aligned} \|z\|^2_{L^2(\Omega;\mathbb{R}^n)} &= \int_\Omega |z|^2 \, dw, \\ |z|^2_{H^1(\Omega;\mathbb{R}^n)} &= \|z\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla z|^2 \, dw, \\ \|z\|^2_{H^1(\Omega;\mathbb{R}^n)} &= \|z\|^2_{L^2(\Omega)} + \|z\|^2_{H_0^1(\Omega)} = \int_\Omega (|z|^2 + |\nabla z|^2) \, dw, \end{aligned}$$

respectively. We let $H_0^1(\Omega;\mathbb{R}^n)$ be the completion of the space of compactly supported smooth functions $C_c^\infty(\Omega;\mathbb{R}^n)$ with respect to the H^1-norm. For all considerations on the unit disc $B \subset \mathbb{R}^3$ we abbreviate

$$\|z\|_{L^2}^2 = \|z\|_{L^2(B;\mathbb{R}^3)}^2,$$
$$|z|_{H^1}^2 = |z|_{H^1(B;\mathbb{R}^3)}^2,$$
$$\|z\|_{H^1}^2 = \|z\|_{H^1(B;\mathbb{R}^3)}^2.$$

We recall that all $z \in H^1(B;\mathbb{R}^3)$ have a well-defined trace $z \mapsto z|_{\partial B} \in L^2(\partial B)$, confer Satz 8.4.3 in [30]. Finally for $1 \leq p < \infty$ we introduce the general L^p-norms

$$\|z\|_{L^p(\Omega;\mathbb{R}^n)}^p = \int_\Omega |z|^p \, dw,$$

and in the case of $p = \infty$ we set

$$\|z\|_{L^\infty(\Omega;\mathbb{R}^n)} = \inf\{C \geq 0 \mid |z(w)| \leq C \text{ for a.e. } w \in \Omega\}.$$

Now let $\Gamma \subset \mathbb{R}^3$ be a closed Jordan curve, i.e. Γ is homeomorphic to the boundary of the disc ∂B. A surface $X \in C^0(\bar{B};\mathbb{R}^3)$ solves Plateau's problem for Γ, if its restriction to B is a minimal surface (Definition 1.1.1), and if it maps ∂B topologically onto Γ.

We have seen in Chapter 1 that minimal surfaces are critical points of the area functional

$$A(X) = \int_B |X_u \wedge X_v| \, dw. \tag{2.1}$$

Therefore, it seems tempting to find minimal surfaces by minimising the area functional over a suitable class of surfaces. Note, however, that the area functional is invariant under changes of parametrisation, i.e.

$$A(X \circ \tau) = A(X) \tag{2.2}$$

for all diffeomorphisms τ of $\bar{B}$. Therefore, any attempt at minimising A is prone to fail, for the invariance of A under any diffeomorphism prevents to distinguish particular parametrisations of a surface X [31].

Instead we consider the Dirichlet integral over B

$$D(X) = \frac{1}{2} \int_B \left(|X_u|^2 + |X_v|^2\right) dw. \tag{2.3}$$

We remark that the Dirichlet functional is invariant under *conformal* diffeomorphisms of $\bar{B}$ [31], i.e. for diffeomorphisms τ such that

$$|\tau_u|^2 = |\tau_v|^2, \qquad \tau_u \cdot \tau_v = 0 \qquad \text{in } \bar{B},$$

there holds

$$D(X \circ \tau) = D(X) \qquad \forall X \in H^1(B; \mathbb{R}^3). \tag{2.4}$$

We derive a relation between A and D as follows. Let $a, b \in \mathbb{R}^3$ be arbitrary vectors. There holds

$$|a \wedge b| \leq |a||b| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2, \tag{2.5}$$

where we used Young's inequality (Lemma 3.5.1 in [28]). Equality holds in (2.5) if and only if a is perpendicular to b and $|a| = |b|$, see [13]. Now suppose $X \in C^1(B; \mathbb{R}^3)$ admits a finite Dirichlet integral (2.3). Equation (2.5) directly implies

$$A(X) \leq D(X) \tag{2.6}$$

with equality if and only if $X_u \cdot X_v = 0$ and $|X_u|^2 = |X_v|^2$ in B. This means that the area functional coincides with Dirichlet's functional on conformally parametrised surfaces [13].

For the converse we have the following statement by Morrey [20].

Theorem 2.1.1. *Let $X \in H^1(B; \mathbb{R}^3)$. Then for every $\varepsilon > 0$ there exists a diffeomorphism $\tau_\varepsilon : B \to B$ such that $Z_\varepsilon := X \circ \tau_\varepsilon$ satisfies*

$$D(Z_\varepsilon) \leq A(Z_\varepsilon) + \varepsilon = A(X) + \varepsilon,$$

where the latter equality follows by invariance of A under change of parametrisation (2.2).

For a proof we refer to Theorem 1.2 in [20]. Before discussing the implications of Morrey's result we will introduce Plateau's problem more rigourously.

Definition 2.1.1 (Disc-type solution to Plateau's problem for Γ)**.** Given a closed Jordan curve $\Gamma \subset \mathbb{R}^3$, the surface $X : \bar{B} \to \mathbb{R}^3$ is a *disc-type solution to Plateau's problem for* Γ if

i. $X \in C^2(B;\mathbb{R}^3) \cap C^0(\bar{B};\mathbb{R}^3)$,

ii. X satisfies a system of nonlinear differential equations

$$\Delta X = 0 \qquad \text{in } B \tag{2.7}$$

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0 \qquad \text{in } B, \tag{2.8}$$

iii. with boundary conditions

$$X|_{\partial B} : \partial B \to \Gamma \text{ is a homeomorphism from } \partial B \text{ onto } \Gamma. \tag{2.9}$$

Condition iii. in Definition 2.1.1 is equivalent to demanding that the restriction $X|_{\partial B}$ is a continuous and strictly monotonic map. Note that uniform limits of strictly monotonic functions may be merely weakly monotonic [13], that is

Definition 2.1.2 (Weakly monotonic map)**.** Let $\Gamma \subset \mathbb{R}^3$ be a closed Jordan curve, let $\varphi : \partial B \to \Gamma$ be a homeomorphism. Then a continuous map $\psi : \partial B \to \Gamma$ is *weakly monotonic*, if there exists an increasing continuous function $\tau : [0, 2\pi] \to \mathbb{R}$ with $\tau(0) = 0, \tau(2\pi) = 2\pi$, such that

$$\psi(e^{i\theta}) = \varphi(e^{i\tau(\theta)}), \quad 0 \leq \theta \leq 2\pi.$$

Intuitively, this means that points w traversing ∂B in a constant direction will be mapped onto image points $\psi(w)$ traversing Γ in a constant direction. With this intuition we understand that weak monotonicity is closed under uniform convergence [13], that is

Lemma 2.1.2. *Assume a sequence $\left(\psi_n\right)_{n\in\mathbb{N}}$ of continuous, weakly monotonic maps from ∂B onto a closed Jordan curve Γ converges uniformly to some map $\psi : \partial B \to \mathbb{R}^3$. Then ψ is a continuous, weakly monotonic map from ∂B onto Γ.*

2.1.2 Variational Formulation

We are now in the position to introduce the class of admissible functions for solutions to Plateau's problem.

Definition 2.1.3 (Class of admissible functions)**.** Let $\Gamma \subset \mathbb{R}^3$ be a closed Jordan curve. The *class of admissible functions* $\mathcal{C}(\Gamma)$ is defined as

$$\begin{aligned}\mathcal{C}(\Gamma) := \{X \in H^1(B;\mathbb{R}^3) \mid X|_{\partial B} &\text{ is represented} \\ &\text{by a continuous, weakly monotonic map } \psi : \partial B \to \Gamma\}.\end{aligned} \tag{2.10}$$

With this definition, we can combine (2.6) with Morrey's result 2.1.1 to infer

$$\inf_{X\in\mathcal{C}(\Gamma)} A(X) = \inf_{X\in\mathcal{C}(\Gamma)} D(X). \tag{2.11}$$

Indeed, $\inf_{X\in\mathcal{C}(\Gamma)} A(X) \leq \inf_{X\in\mathcal{C}(\Gamma)} D(X)$ follows directly by (2.6). Conversely, let $X \in H^1(B;\mathbb{R}^3)$ and let $\varepsilon > 0$. Then there exist diffeomorphisms τ_ε mapping B onto itself, so that $D(X \circ \tau_\varepsilon) \leq A(X) + \varepsilon$. Thus

$$\inf_{X\in\mathcal{C}(\Gamma)} D(X) \leq D(X \circ \tau_\varepsilon) \leq A(X) + \varepsilon.$$

Taking the limit $\varepsilon \to 0$ proves the claim.

Morrey's result justifies why it suffices to minimise Dirichlet's integral for the purpose of minimising the area over $\mathcal{C}(\Gamma)$, for a minimiser of Dirichlet's integral in $\mathcal{C}(\Gamma)$ also minimises the area functional among all surfaces in $\mathcal{C}(\Gamma)$. Therefore the variational problem $\mathcal{P}(\Gamma)$ associated to Plateau's problem for a given curve Γ reads

$$\textit{Minimise } D(X) \textit{ in the class } \mathcal{C}(\Gamma). \tag{$\mathcal{P}(\Gamma)$}$$

Equivalently, we want to find $X_0 \in \mathcal{C}(\Gamma)$ such that

$$D(X_0) = \inf_{X\in\mathcal{C}(\Gamma)} D(X) =: e(\Gamma) \geq 0. \tag{2.12}$$

In order to solve the variational problem ($\mathcal{P}(\Gamma)$) we employ the direct method in the calculus of variations.

2.1.3 Calculus of Variations

This subsection follows Michael Struwe [30]. Let $(H, \|\cdot\|)$ be a Hilbert space with subset $M \subset H$, and consider a functional $F : M \to \mathbb{R}$.

Definition 2.1.4. F is *weakly sequentially lower semi-continuous* at $x_0 \in M$ if there holds

$$\forall \big(x_k\big)_{k\in\mathbb{N}} \subset M : x_k \xrightarrow{w} x_0 \implies F(x_0) \leq \liminf_{k\to\infty} F(x_k), \tag{2.13}$$

where $\xrightarrow{w}$ denotes weak convergence.

Definition 2.1.5. F is *coercive* on M if there holds

$$\forall \big(x_k\big)_{k\in\mathbb{N}} \subset M : \|x_k\| \to \infty \ (k \to \infty) \implies F(x_k) \to \infty \ (k \to \infty). \tag{2.14}$$

With these definitions at hand we can introduce the variational principle.

Theorem 2.1.3 (Variational Principle)**.** *Suppose $M \neq \emptyset$ is a weakly sequentially closed subset of H, and suppose that $F : M \to \mathbb{R}$ is weakly sequentially lower semi-continuous and coercive on M. Then there exists an $x_0 \in M$ with*

$$F(x_0) = \inf_{x\in M} F(x) =: \alpha_0 > -\infty.$$

Proof. Consider a minimising sequence, that is a sequence $\big(x_k\big)_{k\in\mathbb{N}} \subset M$ with $F(x_k) \to \inf_{x\in M} F(x)$, where we note that $\inf_{x\in M} F(x) > -\infty$ since M is assumed to be non-empty. Since F is coercive (2.14), $\big(x_k\big)_{k\in\mathbb{N}}$ is bounded. But H is a Hilbert space, and as such is reflexive. Therefore, by Eberlein–Šmulian's theorem (Satz 5.3.2 in [30]) there exists a weakly convergent subsequence of $\big(x_k\big)_{k\in\mathbb{N}}$, which we again denote by $\big(x_k\big)_{k\in\mathbb{N}}$, with $x_k \xrightarrow{w} x_0$. Since M is weakly sequentially closed, we have that $x_0 \in M$. By weak sequential lower semi-continuity of F (2.13), we conclude for the subsequence $\big(x_k\big)_{k\in\mathbb{N}}$ that

$$\alpha_0 \leq F(x_0) \leq \liminf_{k\to\infty} F(x_k) = \inf_{x\in M} F(x) = \alpha_0.$$

□

2.1.4 Three-Point Condition

Reconsidering the concrete problem ($\mathcal{P}(\Gamma)$), the previous subsection implies that we have to find a minimising sequence whose boundary values contain a uniformly converging subsequence [13]. Note, however, that the set $\mathcal{C}(\Gamma)$ defined in (2.10) is not weakly closed [31]. To recover the requirements of Theorem 2.1.3, we fix three distinct points P_1, P_2, P_3 on ∂B and three distinct points Q_1, Q_2, Q_3 on Γ. We define

$$\mathcal{C}^*(\Gamma) := \{X \in \mathcal{C}(\Gamma) \big| \; X(P_j) = Q_j, \; j = 1, 2, 3\}. \tag{2.15}$$

This is well-defined by the action of the conformal group of the disc on $\mathcal{C}(\Gamma)$, confer Struwe [31]. We set

$$e^*(\Gamma) := \inf_{X \in \mathcal{C}^*(\Gamma)} D(X). \tag{2.16}$$

Since $\mathcal{C}^*(\Gamma) \subset \mathcal{C}(\Gamma)$, we have $e(\Gamma) \leq e^*(\Gamma)$, where $e(\Gamma)$ is as in (2.12). On the other hand, if $X \in \mathcal{C}(\Gamma)$, then there exist three distinct points $\zeta_1, \zeta_2, \zeta_3$ on ∂B such that

$$X(\zeta_j) = Q_j, \; j = 1, 2, 3.$$

Let $\sigma : \bar{B} \to \bar{B}$ be a conformal map such that $\sigma(P_j) = \zeta_j, \; j = 1, 2, 3$. Then $Y := X \circ \sigma \in \mathcal{C}^*(\Gamma)$, and due to the conformal invariance of D (2.4) there holds $D(Y) = D(X)$. Hence we even obtain

$$e(\Gamma) = e^*(\Gamma). \tag{2.17}$$

Consequently, the variational problem

$$\textit{Minimise } D(X) \textit{ in the class } \mathcal{C}^*(\Gamma). \tag{$\mathcal{P}^*(\Gamma)$}$$

is equivalent to ($\mathcal{P}(\Gamma)$). But by imposing the three-point-condition we have gained an important compactness property for the boundary values.

2.1.5 Non-emptiness

The subset M in Theorem 2.1.3 is required to be non-empty. To achieve this, we follow Hildebrandt [13]. We let $\varphi : \partial B \to \Gamma$ be a homeomorphism representing Γ

with Fourier expansion

$$\varphi(e^{i\theta}) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos(k\theta) + B_k \sin(k\theta),$$

where $A_k, B_k \in \mathbb{R}^3$, $k \in \mathbb{N}$. Note that the Fourier expansion converges in $L^2([0, 2\pi]; \mathbb{R}^3)$. Assume φ satisfies the three-point-condition

$$\varphi(P_j) = Q_j, \quad j = 1, 2, 3.$$

Set

$$X(w) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \rho^k \big(A_k \cos(k\theta) + B_k \sin(k\theta)\big),$$

with polar coordinates $w = \rho e^{i\theta}$. Then X is harmonic, $X|_{\partial B} = \varphi$, and

$$\begin{aligned}
D(X) &= \frac{1}{2} \int_B |\nabla X|^2 \, dw \\
&= \frac{1}{2} \int_B \Big| \sum_{k=1}^{\infty} \Big(k\rho^{k-1} (A_k \cos(k\theta) + B_k \sin(k\theta)\big) \\
&\quad + \frac{1}{\rho} \rho^k k(-A_k \sin(k\theta) + B_k \cos(k\theta)\big) \Big) \Big|^2 \, dw \\
&= \frac{1}{2} \int_B \sum_{k=1}^{\infty} k^2 \rho^{2(k-1)} \big(|A_k|^2 + |B_k|^2\big) \, dw \\
&= \frac{1}{2} \sum_{k=1}^{\infty} k^2 \big(|A_k|^2 + |B_k|^2\big) \int_0^{2\pi} \int_0^1 \rho^{2k-1} d\rho d\theta \\
&= \frac{\pi}{2} \sum_{k=1}^{\infty} k \big(|A_k|^2 + |B_k|^2\big).
\end{aligned}$$

As a consequence, we see that X is of class $H^1(B; \mathbb{R}^3)$ if and only if

$$\sum_{k=1}^{\infty} k \big(|A_k|^2 + |B_k|^2\big) < \infty. \tag{2.18}$$

In this case $X \in \mathcal{C}^*(\Gamma)$, i.e. $\mathcal{C}^*(\Gamma) \neq \emptyset$. Now, if $\phi(\theta) := \varphi(e^{i\theta})$ is Lipschitz continuous, then its derivative is bounded and thus also square-integrable. But this implies that (2.18) is satisfied, which in turn implies that $\mathcal{C}^*(\Gamma)$ and $\mathcal{C}(\Gamma)$ are non-empty. We conclude that $\mathcal{C}^*(\Gamma)$ is non-empty if Γ is rectifiable.

2.1.6 Courant–Lebesgue Lemma

There is a final argument missing for the existence proof of Plateau's problem, known as the Courant–Lebesgue Lemma [3].

Lemma 2.1.4 (Courant–Lebesgue). *Let $X \in H^1(B;\mathbb{R}^3)$. For any $w \in \bar{B}$ we denote*

$$C_r = C_r(w) = \bar{B} \cap \partial B_r(w),$$

and we let s be the arc length of C_r. Then for any $0 < \delta < 1$ there exists ρ with $\delta \leq \rho \leq \sqrt{\delta}$ such that $X_s \in L^2(C_\rho)$ and

$$\int_{C_\rho} |X_s|^2 ds \leq \frac{4D(X)}{\rho|\ln \rho|}.$$

The reasoning in the proof follows Struwe [31].

Proof. We can estimate

$$\int_\delta^{\sqrt{\delta}} \int_{C_\rho} |X_s|^2 ds d\rho \leq \int_{\left(B_{\sqrt{\delta}}(w) \setminus B_\delta(w)\right) \cap B} |\nabla X|^2 \, dw \leq 2D(X).$$

Moreover the left hand side can be bounded from below by

$$\operatorname*{ess\,inf}_{\delta \leq \rho \leq \sqrt{\delta}} \left(\rho \int_{C_\rho} |X_s|^2 ds\right) \int_\delta^{\sqrt{\delta}} \frac{1}{\rho} d\rho \leq \int_\delta^{\sqrt{\delta}} \int_{C_\rho} |X_s|^2 ds.$$

But for all ρ with $\delta \leq \rho \leq \sqrt{\delta}$ there holds

$$\int_\delta^{\sqrt{\delta}} \frac{1}{\rho} d\rho = \frac{1}{2}|\ln \delta| \geq \frac{1}{2}|\ln \rho|,$$

Thus we can find $\rho \in [\delta, \sqrt{\delta}]$ such that

$$\int_\delta^{\sqrt{\delta}} \int_{C_\rho} |X_s|^2 ds \leq \frac{4D(X)}{\rho|\ln \rho|}.$$

$\square$

The Courant–Lebesgue Lemma furnishes an argument to prove the following equicontinuity property [31].

Lemma 2.1.5. *Let $X \in \mathcal{C}^*(\Gamma)$ as defined in* (2.15) *with bounded Dirichlet integral $D(X) \leq M$ for some $0 \leq M < \infty$. Let $\varepsilon > 0$ and $w_0 \in \partial B$. Then there exists $\delta > 0$ with $\delta = \delta(\varepsilon, D(X), \Gamma, Q_1, Q_2, Q_3)$ and $Q_j,\ j = 1, 2, 3,$ from* (2.15) *such that for all $w \in \partial B$ there holds*

$$|X(w) - X(w_0)| < 2\varepsilon \tag{2.19}$$

if $|w - w_0| < \delta$.

This statement is equivalent to the equicontinuity of subsets in $\mathcal{C}^*(\Gamma)$ whose Dirichlet integral is uniformly bounded. By the Arzelà–Ascoli theorem (Satz 6.3.1 in [30]) this is equivalent to the compactness of the injection $\mathcal{C}^*(\Gamma) \to C^0(\partial B; \mathbb{R}^3)$. The proof follows the reasoning of Struwe [31].

Proof. Let $\delta_0 > 0$ be sufficiently small, so that any ball of radius $\sqrt{\delta_0}$ contains at most one of the points $P_j,\ j = 1, 2, 3$ from the three-point-condition in (2.15). Choose $\varepsilon_0 > 0$ so that any ball of radius ε_0 contains at most one of the points $Q_j,\ j = 1, 2, 3$. We may assume $\varepsilon < \varepsilon_0$. Then choose ε_1 with $0 < \varepsilon_1 < \varepsilon < \varepsilon_0$ such that for any two points $Y_1, Y_2 \in \Gamma$ contained in a ball of radius ε_1 there is a subarc $\tilde{\Gamma} \subset \Gamma$ with endpoints Y_1 and Y_2 contained in a ball of radius ε. Note that this is possible since Γ is a Jordan curve. By choice of ε_0, for two points Y_1, Y_2 with $|Y_1 - Y_2| < \varepsilon_1$ the subarc $\tilde{\Gamma}$ connecting Y_1 with Y_2 is unique, characterised by the condition that $\tilde{\Gamma}$ contains at most one $Q_j,\ j = 1, 2, 3$.

Choose a maximal δ with $0 < \delta \leq \delta_0$ so that

$$|\ln \delta| \geq \frac{8\pi D(X)}{\varepsilon_1^2}.$$

Choose $\rho \in [\delta, \sqrt{\delta}]$ and $C_\rho(w_0)$ according to the Courant–Lebesgue Lemma 2.1.4 so that

$$\int_{C_\rho} |X_s|^2 ds \leq \frac{4D(X)}{\rho |\ln \rho|}.$$

Let $w_j,\ j = 1, 2$ be the points of intersection of C_ρ with ∂B and denote by $\tilde{C}_\rho = B_\rho(w_0) \cap \partial B$ the subarc of ∂B with endpoints w_1 and w_2. By choice of δ_0, δ and ρ we note that $\tilde{C}_\rho$ contains at most one $P_j,\ j = 1, 2, 3$. Let $X_j = X(w_j),\ j = 1, 2$, and let $\tilde{\Gamma}$ be the subarc of Γ connecting X_1 with X_2. By monotonicity there holds $X(\tilde{C}_\rho) = \tilde{\Gamma}$. Moreover, by choice of C_ρ, with Hölder's inequality (p. 145 in [6]), by Courant–Lebesgue's Lemma and by choice of ρ and δ we find

$$|X_1 - X_2|^2 \le \Big(\int_{C_\rho} |X_s| ds \Big)^2 \le \pi\rho \int_{C_\rho} |X_s|^2 ds \le \frac{4\pi D(X)}{|\ln \rho|} \le \frac{8\pi D(X)}{|\ln \delta|} \le \varepsilon_1^2.$$

Thus we note that $\tilde{\Gamma}$ connecting X_1 with X_2 contains at most one of the points $Q_j,\ j = 1, 2, 3$, and $\tilde{\Gamma}$ is contained in a ball of radius ε. In particular for w_0 and any $w \in \partial B \cap B_\delta(w_0) \subset \tilde{C}_\rho$ there holds

$$|X(w) - X(w_0)| \le 2\varepsilon,$$

since $X(w), X(w_0) \in \tilde{\Gamma}$ are both contained in a ball of radius ε. But this yields the claim, for $\delta = \delta(D(X), \varepsilon_1)$ and $\varepsilon_1 = \varepsilon_1(D(X), \varepsilon, \Gamma, Q_1, Q_2, Q_3)$. $\square$

This equicontinuity property now yields weak closedness of $\mathcal{C}^*(\Gamma)$.

Lemma 2.1.6. *$\mathcal{C}^*(\Gamma)$ as defined in (2.15) is closed with respect to the weak topology in $H^1(B; \mathbb{R}^3)$.*

Proof. Let $\big(X_n\big)_{n\in\mathbb{N}} \subset \mathcal{C}^*(\Gamma)$ be such that $X_n \xrightarrow{w} X$ converges weakly in $H^1(B; \mathbb{R}^3)$. By weak convergence, $\big(X_n\big)_{n\in\mathbb{N}}$ is bounded in $H^1(B; \mathbb{R}^3)$ (Satz 4.6.1 in [30]); in particular,

$$D(X_n) \le M$$

uniformly in n for some $M \in \mathbb{R}$. Lemma 2.1.5 implies uniform convergence of a subsequence $X_n \to X$ on ∂B. Thus $X \in C^0(\partial B) \cap H^1(B; \mathbb{R}^3)$, satisfies the three-point condition, and by Lemma 2.1.2 X maps ∂B weakly monotonically onto Γ. Thus $X \in \mathcal{C}^*(\Gamma)$. $\square$

2.1.7 Existence

We now state the existence theorem [13] due to Douglas and Radò.

Theorem 2.1.7. *Let $\Gamma \subset \mathbb{R}^3$ be a closed Jordan curve. If $\mathcal{C}(\Gamma)$ defined in* (2.10) *is nonempty, then the variational problem $(\mathcal{P}(\Gamma))$ has at least one solution. In particular, $(\mathcal{P}(\Gamma))$ has a solution for every rectifiable curve Γ.*

Proof. We will need the following two statements for the proof.

Claim 2.1.7.1. The functional D (2.3) is coercive (2.14) on $\mathcal{C}(\Gamma)$.

Proof of Claim 2.1.7.1. Let $X \in \mathcal{C}(\Gamma)$. Consider a harmonic $Y \in H^1(B;\mathbb{R}^3) \cap L^\infty(\partial B;\mathbb{R}^3)$ satisfying $Y|_{\partial B} = X|_{\partial B}$. Then there holds $X - Y \in H^1_0(B;\mathbb{R}^3)$. Thus we can estimate with Poincaré's inequality [6]

$$\|X - Y\|^2_{L^2(B;\mathbb{R}^3)} \leq C\|\nabla(X - Y)\|^2_{L^2(B;\mathbb{R}^3)}. \tag{2.20}$$

Note that integration by parts gives by choice of Y

$$\begin{aligned}
\int_B |\nabla(X - Y)|^2 \, dw &= -\int_B (X - Y)\Delta X \, dw \\
&= \int_B \nabla(X - Y)\nabla X \, dw \\
&= -\int_B \Delta X \cdot X \, dw \\
&= \int_B |\nabla X|^2 \, dw.
\end{aligned}$$

Therefore by the triangle inequality and by (2.20) we obtain

$$\|X\|^2_{L^2(B;\mathbb{R}^3)} - \|Y\|^2_{L^2(B;\mathbb{R}^3)} \leq \|X - Y\|^2_{L^2(B;\mathbb{R}^3)} \leq C\|\nabla X\|^2_{L^2(B;\mathbb{R}^3)}.$$

We can rearrange and apply Hölder's inequality and the maximum principle for harmonic functions [19] to find

$$\begin{aligned}
\|X\|^2_{L^2(B;\mathbb{R}^3)} &\leq C\|\nabla X\|^2_{L^2(B;\mathbb{R}^3)} + C\|Y\|^2_{L^\infty(B;\mathbb{R}^3)} \\
&= C\|\nabla X\|^2_{L^2(B;\mathbb{R}^3)} + C\|Y\|^2_{L^\infty(\partial B;\mathbb{R}^3)} \\
&= C\|\nabla X\|^2_{L^2(B;\mathbb{R}^3)} + C\|X\|^2_{L^\infty(\partial B;\mathbb{R}^3)},
\end{aligned}$$

where the last equality is due to the choice of Y. But by definition of $\mathcal{C}(\Gamma)$ (2.10) there holds $X|_{\partial B} \in C^0(\partial B; \mathbb{R}^3)$. Thus in view of (2.3) we conclude

$$\begin{aligned} \|X\|^2_{H^1(B;\mathbb{R}^3)} &= \|X\|^2_{L^2(B;\mathbb{R}^3)} + \|\nabla X\|^2_{L^2(B;\mathbb{R}^3)} \\ &\leq C\|\nabla X\|^2_{L^2(B;\mathbb{R}^3)} + C\|X\|^2_{L^\infty(\partial B;\mathbb{R}^3)} \\ &\leq CD(X). \end{aligned}$$

□

Moreover we claim

Claim 2.1.7.2. The functional D (2.3) is weakly sequentially lower semi-continuous (2.13) on $H^1(B; \mathbb{R}^3)$.

Proof of Claim 2.1.7.2. We write for $X, Y \in H^1(B; \mathbb{R}^3)$

$$D(X, Y) := \frac{1}{2} \int_B \nabla X \cdot \nabla Y \, dw,$$

so that $D(X, X) = D(X)$ as defined in (2.3). Then we note that D is a bilinear functional on the Hilbert space $H^1(B; \mathbb{R}^3)$. Thus we find for a weakly convergent sequence $X_n \xrightarrow{w} X$,

$$0 \leq D(X_n - X) = D(X_n - X, X_n - X) = D(X_n) - D(X) - 2D(X, X_n - X).$$

But $D(X, \cdot)$ is a continuous linear functional; thus by weak convergence of $\left(X_n\right)_{n\in\mathbb{N}}$, we find $D(X, X_n - X) \to 0$ as $(n \to \infty)$. This proves the weak sequential lower semi-continuity of D. □

Now note that $\mathcal{C}^*(\Gamma) \subset H^1(B; \mathbb{R}^3)$ is non-empty, either by assumption, or else since Γ is rectifiable, see Section 2.1.5. Moreover $\mathcal{C}^*(\Gamma)$ is weakly closed in $H^1(B; \mathbb{R}^3)$ by Lemma 2.1.6. Finally, Claim 2.1.7.1 implies in particular that D is coercive on $\mathcal{C}^*(\Gamma)$, and Claim 2.1.7.2 yields weak sequential lower semi-continuity of D on $\mathcal{C}^*(\Gamma)$. Therefore Theorem 2.1.3 applies and we obtain that there exists $X \in C^*(\Gamma)$ such that

$$D(X) = e^*(\Gamma).$$

Finally, note that by (2.17) we have $e^*(\Gamma) = e(\Gamma)$, i.e. X solves $(\mathcal{P}(\Gamma))$. □

2.1.8 Harmonicity

We now derive harmonicity of solutions X of $(\mathcal{P}(\Gamma))$ by means of *variations of the surface* or *outer variations* following Struwe [31].

Lemma 2.1.8. *Let $X \in \mathcal{C}(\Gamma)$ (2.10). Then there holds for the Dirichlet functional* (2.3)

$$\frac{d}{d\varepsilon} D(X + \varepsilon\varphi)\Big|_{\varepsilon=0} = 0 \qquad \forall \varphi \in H_0^1(B; \mathbb{R}^3), \tag{2.21}$$

if and only if

$$\Delta X = 0 \qquad \text{in } B. \tag{2.22}$$

Proof. We compute for all $\varphi \in H_0^1(B; \mathbb{R}^3)$

$$\frac{d}{d\varepsilon} D(X + \varepsilon\varphi)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\Big(\frac{1}{2}\int_B |\nabla X + \varepsilon \nabla\varphi|^2 \, dw\Big)\Big|_{\varepsilon=0} = \int_B \nabla X \nabla \varphi \, dw. \tag{2.23}$$

Thus if (2.21) holds, then

$$0 = \int_B \nabla X \nabla \varphi \, dw, \quad \forall \varphi \in H_0^1(B; \mathbb{R}^3).$$

Hence, X weakly solves (2.22). By Weyl's Lemma (Lemma 2 in [35]), $X \in C^2(B; \mathbb{R}^3)$ and (2.22) holds in a classical sense.

Conversely, if $\Delta X = 0$, then for all $\varphi \in H_0^1(B; \mathbb{R}^3)$

$$0 = -\int_B \Delta X \varphi \, dw = \int_B \nabla X \nabla \varphi \, dw = \frac{d}{d\varepsilon} D(X + \varepsilon\varphi)\Big|_{\varepsilon=0}$$

where we integrated by parts and used (2.23). □

2.1.9 Conformality

We finally want to show that any solution of the variational problem $(\mathcal{P}(\Gamma))$ satisfies the conformality relations (2.8). To this end we introduce the technicalities for *variations of the parametrisation of X* or *inner variations*. We follow Hildebrandt's explanations [13].

Let $\lambda = (\mu, \nu)$ be an arbitrary vector field on $\bar{B}$ of class $C^1(\bar{B}; \mathbb{R}^2)$. We may assume that λ is defined on all of $\mathbb{R}^2$, so that $\lambda \in C^1(\mathbb{R}^2; \mathbb{R}^2)$. We introduce a one-parameter family $\tau_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$\tau_\varepsilon(w) = \tau(w, \varepsilon) = w - \varepsilon\lambda(w).$$

Then $\tau \in C^1(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2)$. Let B_0 be some open set compactly containing B, a relation which we denote by $B \subset\subset B_0$. Since τ_ε is just a perturbation of the identity, we see that $\tau_\varepsilon : B_0 \to \tau_\varepsilon(B_0)$ maps B_0 diffeomorphically onto $\tau_\varepsilon(B_0)$ if $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ for sufficiently small $\varepsilon_0 > 0$. Therefore, the inverse $\sigma_\varepsilon := \tau_\varepsilon^{-1}$ exists on some domain Ω where $\tau_\varepsilon(B) \subset\subset \Omega \subset\subset B_0$. To be precise, we let $\omega = \tau_\varepsilon(w) = \tau(w, \varepsilon)$ and $w = \sigma_\varepsilon(\omega) = \sigma(\omega, \varepsilon)$. Then $\sigma \in C^1(\Omega \times (-\varepsilon_0, \varepsilon_0); \bar{B})$ and there holds

$$\sigma(\omega, \varepsilon) = \omega + \varepsilon\lambda(\omega), \quad \tau(\sigma(\omega, \varepsilon), \varepsilon) = \omega, \quad \forall(\omega, \varepsilon) \in \Omega \times (-\varepsilon_0, \varepsilon_0).$$

Restricting τ_ε to $\bar{B}$ and σ_ε to $\tau_\varepsilon(\bar{B})$ we obtain a diffeomorphism $\tau_\varepsilon : \bar{B} \to \tau_\varepsilon(\bar{B})$ with inverse σ_ε satisfying

$$\tau_0(B) = B, \quad \sigma_0(w) = w, \quad \frac{\partial}{\partial\varepsilon}\sigma(w, \varepsilon)\big|_{\varepsilon=0} = \lambda(w) \quad \text{for } w \in \bar{B}.$$

Moreover, we have

$$\nabla\tau_\varepsilon = \mathrm{id} - \varepsilon\nabla\lambda,$$

and thus

$$\det(\nabla\tau_\varepsilon) = 1 - \varepsilon\big(\mu_u + \nu_v\big). \tag{2.24}$$

Now let $X \in C^1(\bar{B}; \mathbb{R}^3)$, and define $Z_\varepsilon := X \circ \sigma_\varepsilon : \tau_\varepsilon(\bar{B}) \to \mathbb{R}^3$. To emphasise the domain, we denote the Dirichlet integral over B with D_B. We define

Definition 2.1.6 (First inner variation)**.** The *first inner variation* of the Dirichlet integral D_B at X in direction of $\lambda = (\mu, \nu) \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ is defined as

$$\partial D_B(X, \lambda) := \frac{d}{d\varepsilon} D_{\tau_\varepsilon(B)}(X \circ \sigma_\varepsilon)\big|_{\varepsilon=0},$$

where $D_{\tau_\varepsilon(B)}$ represents the Dirichlet integral over $\tau_\varepsilon(B)$.

Based on Definition 2.1.6 we find for any $\lambda = (\mu, \nu) \in C^1(\bar{B}; \mathbb{R}^2)$

$$\begin{aligned}\partial D_B(X,\lambda) &= \frac{d}{d\varepsilon} D_{\tau_\varepsilon(B)}(X \circ \sigma_\varepsilon)\Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\Big(\frac{1}{2}\int_{\tau_\varepsilon(B)} |\nabla(X \circ \sigma_\varepsilon)|^2 d\omega\Big)\Big|_{\varepsilon=0} \\ &= \frac{1}{2}\frac{d}{d\varepsilon}\int_{\tau_\varepsilon(B)} |\big(\nabla(X) \circ \sigma_\varepsilon\big) \cdot \nabla\sigma_\varepsilon|^2 d\omega\Big|_{\varepsilon=0} \\ &= \frac{1}{2}\frac{d}{d\varepsilon}\int_B |\nabla X \cdot \big((\nabla\sigma_\varepsilon) \circ \tau_\varepsilon\big)|^2 \det(\nabla\tau_\varepsilon)\, dw\Big|_{\varepsilon=0}.\end{aligned} \tag{2.25}$$

Moreover there holds by the chain rule

$$\mathrm{id} = \nabla(\mathrm{id}) = \nabla(\sigma_\varepsilon \circ \tau_\varepsilon) = \big((\nabla\sigma_\varepsilon) \circ \tau_\varepsilon\big) \circ \nabla\ \tau_\varepsilon,$$

and thus

$$\big((\nabla\sigma_\varepsilon) \circ \tau_\varepsilon\big) = (\nabla\ \tau_\varepsilon)^{-1} = (\mathrm{id} - \varepsilon\nabla\lambda)^{-1}.$$

But this inverse is given by the Neumann series

$$\big((\nabla\sigma_\varepsilon) \circ \tau_\varepsilon\big) = (\mathrm{id} - \varepsilon\nabla\lambda)^{-1} = \sum_{k=0}^{\infty} \big(\varepsilon\nabla\lambda\big)^k \tag{2.26}$$

which converges for small ε, see Beispiel 2.2.2 ii) in [30]. Therefore we obtain from (2.25) with (2.24) and (2.26)

$$\begin{aligned}&\partial D_B(X,\lambda) \\ &= \frac{1}{2}\frac{d}{d\varepsilon}\int_B \big(|\nabla X|^2 + 2\varepsilon(\nabla X)^T\nabla\lambda\nabla X + \mathcal{O}(\varepsilon^2)\big)\det(\nabla\tau_\varepsilon)\, dw\Big|_{\varepsilon=0} \\ &= \frac{1}{2}\int_B \big(2(\nabla X)^T \cdot \nabla\lambda \cdot \nabla X - |\nabla X|^2\big(\mu_u + \nu_v\big)\big)\, dw \\ &= \int_B \big(|X_u|^2\mu_u + |X_v|^2\nu_v + X_u \cdot X_v\big(\mu_v + \nu_u\big)\big)\, dw \\ &\quad - \frac{1}{2}\int_B \big(|X_u|^2\mu_u + |X_u|^2\nu_v + |X_v|^2\mu_u + |X_v|^2\nu_v\big)\, dw \\ &= \frac{1}{2}\int_B \big(|X_u|^2 - |X_v|^2\big)\big(\mu_u - \nu_v\big) + 2X_u \cdot X_v\big(\mu_v + \nu_u\big)\, dw.\end{aligned} \tag{2.27}$$

With this discussion we arrive at the following lemma.

Lemma 2.1.9. *Let $X \in H^1(B;\mathbb{R}^3)$. Then there holds for the first variation of X as defined in Definition 2.1.6*

$$\partial D_B(X,\lambda) = 0 \qquad \textit{for all } \lambda \in C^1(\bar{B};\mathbb{R}^2), \tag{2.28}$$

if and only if X is conformal, i.e.

$$|X_u|^2 = |X_v|^2, \qquad X_u \cdot X_v = 0 \quad \textit{in} \;\; B. \tag{2.29}$$

Proof. Assume first (2.28) holds. Choose arbitrary $\zeta, \eta \in C_c^\infty(B)$ and determine functions $f, g \in C^\infty(\bar{B})$ solving

$$\Delta f = \zeta,\;\; \Delta g = \eta \;\; \text{in } B, \quad f = 0,\;\; g = 0 \;\; \text{on } \partial B.$$

Then $\mu := f_u + g_v,\;\; \nu = -f_v + g_u$ are of class $C^\infty(\bar{B})$, and satisfy

$$\mu_u - \nu_v = \zeta, \quad \mu_v + \nu_u = \eta.$$

Testing (2.28) with this particular choice of $\lambda = (\mu,\nu)$ we see with (2.27)

$$\begin{aligned} 0 &= \partial D_B(X,\lambda) \\ &= \frac{1}{2}\int_B \big(|X_u|^2 - |X_v|^2\big)\big(\mu_u - \nu_v\big) + 2X_u \cdot X_v\big(\mu_v + \nu_u\big)\, dw \\ &= \frac{1}{2}\int_B \big(|X_u|^2 - |X_v|^2\big)\zeta + 2X_u \cdot X_v \;\eta\, dw. \end{aligned}$$

But since ζ, η were arbitrarily chosen, we infer that $|X_u|^2 - |X_v|^2 = 0$, and $X_u \cdot X_v = 0$ almost everywhere in B.

Conversely, assume X satisfies the conformality relations (2.29). But (2.27) implies for all $\mu, \nu \in C^1(\bar{B})$

$$0 = \frac{1}{2}\int_B \big(|X_u|^2 - |X_v|^2\big)\big(\mu_u - \nu_v\big) + 2X_u \cdot X_v\big(\mu_v + \nu_u\big)\, dw = \partial D_B(X,\lambda),$$

where $\lambda = (\mu,\nu)$. $\square$

By aid of this lemma we conclude

Lemma 2.1.10. *Every solution X of* $(\mathcal{P}(\Gamma))$ *is conformal* (2.8) *in B.*

Proof. Consider $\sigma_\varepsilon : \tau_\varepsilon(\bar{B}) \to \bar{B}$ as above. Since B and $\tau_\varepsilon(B)$ are diffeomorphic, Riemann's mapping theorem [1] implies that there exists a conformal map $\kappa_\varepsilon : B \to \tau_\varepsilon(B)$. By Caratheodory's theorem [5] κ_ε can be extended to a homeomorphism from $\bar{B}$ onto $\tau_\varepsilon(\bar{B})$. Define $Z_\varepsilon := X \circ \sigma_\varepsilon : \tau_\varepsilon(\bar{B}) \to \mathbb{R}^3$, and $Y_\varepsilon := Z_\varepsilon \circ \kappa_\varepsilon : \bar{B} \to \mathbb{R}^3$. By conformal invariance of Dirichlet's integral (2.4), we have

$$D_B(Y_\varepsilon) = D_{\tau_\varepsilon(B)}(Z_\varepsilon).$$

Now assume X solves $(\mathcal{P}(\Gamma))$. Then $X \in \mathcal{C}(\Gamma)$, and X minimises Dirichlet's integral (2.3) in B. Hence

$$D_B(X) \leq D_B(Y_\varepsilon) = D_{\tau_\varepsilon(B)}(Z_\varepsilon).$$

Therefore, $\partial D_B(X, \lambda) = 0$ for all $\lambda \in C^1(\bar{B}; \mathbb{R}^2)$. By Lemma 2.1.9 we thus obtain that X satisfies the conformality relations (2.8). $\square$

2.1.10 Plateau's Boundary Condition

We derive that solutions to Plateau's problem map ∂B topologically onto Γ. We closely follow Hildebrandt [13].

Lemma 2.1.11. *Every conformal* (2.8) *minimal surface X of class $\mathcal{C}(\Gamma)$ maps ∂B topologically onto Γ.*

Proof. Since $X|_{\partial B}$ is weakly monotonic, it suffices to prove that $X|_{\partial B}$ is injective. Suppose by contradiction that this does not hold. Then we could find an arc $C = \{e^{i\theta} \mid \theta_1 < \theta < \theta_2\}$ for some $\theta_1, \theta_2 \in \mathbb{R}$, such that X maps C onto a single point $P \in \mathbb{R}^3$, i.e.

$$X(C) = P.$$

By Schwarz's reflection principle [1] we can extend X to a harmonic map across

C. Differentiating X in tangential direction would then yield

$$\frac{d}{d\theta} X(e^{i\theta}) = 0,$$

and thus by conformality of X we would then find $\nabla X \equiv 0$ on C. This implies $\nabla X \equiv 0$ in B. Hence $X \equiv P$ in B, contradicting $X \in \mathcal{C}(\Gamma)$. $\square$

2.1.11 Solution of Plateau's Problem

We can now combine the previous results to formulate the main existence theorem.

Theorem 2.1.12. *Let $\Gamma \subset \mathbb{R}^3$ be a closed Jordan curve. Assume $\mathcal{C}(\Gamma)$ as defined in* (2.10) *is non-empty. Then the minimisation problem* ($\mathcal{P}(\Gamma)$) *has at least one solution. Moreover, every surface X solving* ($\mathcal{P}(\Gamma)$) *satisfies*

i. $X \in C^2(B) \cap C^0(\bar{B})$,

ii. *X is harmonic* (2.7) *and conformal* (2.8) *in B,*

iii. *X maps ∂B topologically onto Γ* (2.9).

In particular, any closed rectifiable curve $\Gamma \subset \mathbb{R}^3$ bounds at least one disc-type minimal surface, see Definition 2.1.1.

Proof. Firstly there exists at least one solution by Theorem 2.1.7. Therefore, assume $X \in \mathcal{C}(\Gamma)$ solves ($\mathcal{P}(\Gamma)$).

Since X minimises the Dirichlet integral, equation (2.21) is satisfied, and thus by Lemma 2.1.8 $\Delta X = 0$ weakly. But with Weyl's Lemma (Lemma 2 in [35]), we even find that $X \in C^2(B; \mathbb{R}^3)$ and X is harmonic in a classical sense. Moreover by definition of $\mathcal{C}(\Gamma)$ (2.10) there holds $X|_{\partial B} \in C^0(\partial B; \mathbb{R}^3)$. Thus with the maximum principle for harmonic functions [19] we infer $X \in C^0(\bar{B}; \mathbb{R}^3)$. Therefore we conclude $X \in C^2(B; \mathbb{R}^3) \cap C^0(\bar{B}; \mathbb{R}^3)$.

By Lemma 2.1.10 X satisfies the conformality relations (2.8) in B.

Finally by Lemma 2.1.11 we even obtain that X satisfies (2.9), i.e. X is of the type of the disc. $\square$

Let us remark that by (2.11) any solution X of the minimisation problem $(\mathcal{P}(\Gamma))$ is a surface of least area in $\mathcal{C}(\Gamma)$.

2.1.12 A-priori Bound

As a final remark, we state the isoperimetric inequality for minimal surfaces Theorem III.3.5 in Courant [3].

Theorem 2.1.13. *Let $\Gamma \subset \mathbb{R}^3$ be a rectifiable Jordan curve of finite length $L(\Gamma) < \infty$. Then any $X \in \mathcal{C}(\Gamma)$ (2.10) solving (2.7)–(2.9) satisfies the estimate*

$$D(X) \leq \frac{(L(\Gamma))^2}{4\pi}, \tag{2.30}$$

where D is the Dirichlet integral (2.3).

Equation (2.30) implies that minimal surfaces bounded by rectifiable Jordan curves admit a finite Dirichlet integral. A nice proof can be found in do Carmo [2].

2.2 H-Surfaces

In the previous section we studied surfaces X bounded by a prescribed Jordan curve Γ in $\mathbb{R}^3$ whose mean curvature vanishes. It is now natural to ask: upon prescribing in addition to a Jordan curve a value $H \neq 0$ for the mean curvature, are there are surfaces X bounded by Γ with mean curvature H? Such surfaces, we will call H-surfaces.

Throughout Section 2.2 we consider a Jordan curve $\Gamma \subset \mathbb{R}^3$, and we let $H \in \mathbb{R}$ be some constant value. Again we can parametrise the problem by introducing isothermal coordinates over the disc. Determine a surface $X : \bar{B} \to \mathbb{R}^3$ of the form

$$X(u, v) = (X^1(u, v), X^2(u, v), X^3(u, v)),$$

satisfying a system of nonlinear differential equations

$$\Delta X = 2H X_u \wedge X_v \qquad \text{in } B \tag{2.31}$$

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0 \qquad \text{in } B \tag{2.32}$$

$$X|_{\partial B} : \partial B \to \Gamma \text{ is a homeomorphism of } \partial B \text{ onto } \Gamma. \tag{2.33}$$

A surface X solving (2.31)–(2.33) arises as a soap bubble, i.e. as a surface of least area enclosing a given volume. We may restrict all considerations to curves Γ contained in the unit ball around the origin in $\mathbb{R}^3$.

Remark 1. We may recognise equation (2.31) formally as the Euler–Lagrange equations associated to the functional

$$E_H(X) := \frac{1}{2}\int_B |\nabla X|^2 \, dw + \frac{2H}{3}\int_B X \cdot X_u \wedge X_v \, dw = D(X) + 2HV(X) \tag{2.34}$$

for $X \in C^2(B; \mathbb{R}^3)$, where D is the Dirichlet integral as introduced in (2.3), and V is the volume integral over B

$$V(X) := \frac{1}{3}\int_B X \cdot X_u \wedge X_v \, dw. \tag{2.35}$$

Indeed, we compute for $\varphi \in C_0^\infty(B; \mathbb{R}^3)$

$$\begin{aligned}
&\frac{d}{d\varepsilon} E_H(X + \varepsilon\varphi)\Big|_{\varepsilon=0} \\
&\quad = \int_B \nabla X \cdot \nabla\varphi \, dw + \frac{2H}{3}\int_B X_u \wedge X_v \cdot \varphi + (X_u \wedge \varphi_v + \varphi_u \wedge X_v) \cdot X \, dw \\
&\quad = -\int_B \Delta X \cdot \varphi \, dw + \frac{2H}{3}\int_B X_u \wedge X_v \cdot \varphi - (X_u \wedge \varphi \cdot X_v + \varphi \wedge X_v \cdot X_u) \, dw \\
&\quad = -\int_B \Delta X \cdot \varphi \, dw + \frac{2H}{3}\int_B X_u \wedge X_v \cdot \varphi + (X_u \wedge X_v + X_u \wedge X_v) \cdot \varphi \, dw \\
&\quad = -\int_B \Delta X \cdot \varphi \, dw + 2H\int_B X_u \wedge X_v \cdot \varphi \, dw,
\end{aligned}$$

where we integrated by parts for the second equality, and used antisymmetry of the exterior product in the third equality. Note that all boundary terms arising with the integration by parts vanish due to the compact support of φ. Therefore we see that critical points $X \in C^2(B; \mathbb{R}^3)$ for E_H solve (2.31).

We refrain from going into details of another existence proof, yet we want to emphasise some properties of the functional under consideration [31].

Remark 2. i. $V(X)$ is well-defined and trilinear on $H^1 \cap L^\infty(B;\mathbb{R}^3)$.

ii. V is invariant under orientation preserving reparametrisations of X. Indeed, let $X \in H^1 \cap L^\infty(B;\mathbb{R}^3)$, and let $\tau \in C^1(\bar{B};\mathbb{R}^2)$ be a diffeomorphism of B onto $\tau(B)$ with $\det(\nabla\tau) > 0$. Define $\sigma := \tau^{-1}$ from $\tau(B)$ onto B, and let $Y := X \circ \sigma \in H^1 \cap L^\infty(\tau(B);\mathbb{R}^3)$. Then

$$\begin{aligned} V(Y) &= \frac{1}{3}\int_{\tau(B)} Y_u \wedge Y_v \cdot Y \, dw \\ &= \frac{1}{3}\int_B X_u \wedge X_v \cdot X \det(\nabla\sigma \circ \tau)|\det(\nabla\tau)| \, dw \\ &= \frac{1}{3}\int_B X_u \wedge X_v \cdot X \det(\nabla\sigma \circ \tau)\det(\nabla\tau) \, dw \\ &= \frac{1}{3}\int_B X_u \wedge X_v \cdot X \det((\nabla\sigma \circ \tau)\nabla\tau) \, dw \\ &= V(X), \end{aligned} \tag{2.36}$$

where we used the non-negativity of $\det(\nabla\tau)$, multiplicativity of the determinant, and the observation based on the chain rule

$$\mathrm{id} = \nabla\mathrm{id} = \nabla(\sigma \circ \tau) = (\nabla\sigma \circ \tau)\nabla\tau.$$

iii. Let $X \in \mathcal{C}(\Gamma) \cap C^2(B;\mathbb{R}^3)$, where $\mathcal{C}(\Gamma)$ is as in (2.10). Then by Remark 1 we derive the weak formulation of (2.31)

$$\begin{aligned} \frac{d}{d\varepsilon} E_H(X + \varepsilon\varphi)\Big|_{\varepsilon=0} &= \int_B \nabla X \nabla\varphi \, dw + 2H \int_B X_u \wedge X_v \cdot \varphi \, dw \\ &= \int_B \big(-\Delta X + 2H X_u \wedge X_v \big)\varphi \, dw = 0 \end{aligned} \tag{2.37}$$

for all $\varphi \in H_0^1(B;\mathbb{R}^3)$. Moreover, for the weak form of the conformality relations (2.32) we find by (2.36)

$$\partial E_H(X,\lambda) = \partial D_B(X,\lambda) = 0 \quad \forall \lambda \in C^1(\bar{B};\mathbb{R}^2). \tag{2.38}$$

Remark 2 iii. means that formally we have that stationarity in the inner variations and outer variations corresponds to solutions of (2.31) satisfying (2.32). However, we had to assume a-priori sufficient regularity for the surface. To obtain the analogue of Lemma 2.1.8 and Lemma 2.1.10 for H-surfaces, we first have to derive some continuity properties of the volume functional.

2.2.1 The Volume Functional

Extending the volume functional continuously onto $\mathcal{C}(\Gamma)$ rests on the following inequality (Theorem 2.5 in [23]).

Theorem 2.2.1. *Let $X, Y \in H^1 \cap L^\infty(B;\mathbb{R}^3)$ with $X - Y \in H_0^1(B;\mathbb{R}^3)$. Then*

$$|V(X) - V(Y)|^2 \leq \frac{\big(D(X) + D(Y)\big)^3}{36\pi} \tag{2.39}$$

for D as in (2.3) *and V as in* (2.35).

Firstly note that combining Morrey's result on ε-conformality 2.1.1 with (2.39) and with the invariance of V under orientation-preserving reparametrisations (2.36) yields

$$|V(X) - V(Y)|^2 \leq \frac{\big(A(X) + A(Y)\big)^3}{36\pi},$$

for all $X, Y \in H^1 \cap L^\infty(B;\mathbb{R}^3)$ with the property that there exists an oriented diffeomorphism $\tau : \bar{B} \to \bar{B}$ such that $X|_{\partial B} = Y \circ \tau|_{\partial B}$, see [31].

Secondly, Theorem 2.2.1 implies that the volume functional V is a continuous functional in the norm topology of $H_0^1(B;\mathbb{R}^3)$. However, V is not continuous with respect to the weak topology on $H_0^1(B;\mathbb{R}^3)$, confer [34].

Due to inequality (2.39) we can establish that the volume functional continuously extends to $H_0^1(B;\mathbb{R}^3)$. In a further step, we can also extend V onto $\mathcal{C}(\Gamma)$, confer Wente's approach in [34]. In particular, with Remark 2 iii. we thus arrive at the following analogue of Lemmata 2.1.8 and 2.1.10.

Lemma 2.2.2. *Let $H \in \mathbb{R}$ and let $\Gamma \subset B \subset \mathbb{R}^3$ be a closed Jordan curve. Let $X \in \mathcal{C}(\Gamma)$ as defined in* (2.10). *Then X is conformal* (2.32) *and weakly solves*

(2.31) *if and only if*

$$\frac{d}{d\varepsilon}E_H(X+\varepsilon\varphi)\Big|_{\varepsilon=0} = 0 \qquad \forall\varphi \in H_0^1(B;\mathbb{R}^3),$$
$$\partial E_H(X,\lambda) = 0 \qquad \forall\lambda \in C^1(\bar{B};\mathbb{R}^2).$$

2.2.2 Existence

With Lemma 2.2.2 we infer that minimisers of E_H in $\mathcal{C}(\Gamma)$ weakly solve (2.31)–(2.32). Let us denote by $\mathcal{P}_H(\Gamma)$ the minimisation problem

$$\textit{Minimise } E_H(X) \textit{ in the class } \mathcal{C}(\Gamma). \tag{$\mathcal{P}_H(\Gamma)$}$$

The existence of such minimisers has been derived under various geometrical assumptions on Γ and H. Notably, Wente illustrated the existence of solutions to $\mathcal{P}_H(\Gamma)$ in [34] provided that H satisfies $|H|\sqrt{\alpha_\Gamma} < \frac{\sqrt{\pi}}{5}$, where $\alpha_\Gamma := \inf_{X\in\mathcal{C}(\Gamma)} A(X)$. On the other hand, Hildebrandt obtained solutions to $\mathcal{P}_H(\Gamma)$ whenever $\Gamma \subset B_R(0) \subset \mathbb{R}^3$ and H satisfies $|H|R \leq 1$, see [14]. Here we denote by $B_R(0)$ the ball of radius R around the origin

$$B_R(0) = \{z \in \mathbb{R}^3 |\ \|z\| < R\}.$$

The following example based on Wente [34] demonstrates the differences between these results.

Example 2.2.1. i. Assume Γ bounds a rectangle of length a and width b for some $a, b \in \mathbb{R}$. Then $\alpha_\Gamma = ab$, and thus Wente's result [34] yields existence of solutions to ($\mathcal{P}_H(\Gamma)$) whenever

$$|H| < \frac{\sqrt{\pi}}{5ab}.$$

On the other hand, we note that the rectangle Γ lies in a ball of radius $R = \frac{\sqrt{a^2+b^2}}{2}$ by a simple geometrical consideration. Thus for this case Hildebrandt [14] establishes existence provided that

$$|H| \leq \frac{2}{\sqrt{a^2+b^2}}.$$

Thus, depending on the ratio of length to width, either Wente's or Hildebrandt's result yields better bounds.

ii. Now assume $\Gamma = \partial B_1(0)$. In this case, $\alpha_\Gamma = \pi$, and by Wente's result solutions exist whenever
$$|H| < \frac{1}{5}.$$
On the other hand, with Hildebrandt we obtain
$$|H| \leq 1.$$
In particular, Heinz has demonstrated that for planar circles Hildebrandt's result cannot be improved [8].

However, let us now consider a distorted circle of contour Γ_ε, obtained by cutting off an arc of length ε^2 and inserting a spike of height $\frac{2}{\varepsilon}$. Then $\alpha_\Gamma \approx \pi + \frac{1}{2}\frac{2\varepsilon^2}{\varepsilon} = \pi + \varepsilon$, and thus Wente yields existence for
$$|H| < \frac{\sqrt{\pi}}{5\sqrt{\pi+\varepsilon}} \to \frac{1}{5} \quad (\varepsilon \to 0).$$
On the other hand, Γ_ε is contained in a ball of radius $R = 1 + \frac{1}{\varepsilon}$. Thus with Hildebrandt we have solutions whenever
$$|H| \leq \frac{\varepsilon}{\big(\varepsilon+1\big)} \to 0 \quad (\varepsilon \to 0).$$

We conclude the discussion on the existence of H-surfaces by the remark that Hildebrandt's result can be obtained with similar variational methods as for the classical Plateau problem, where we first introduce a suitable class of admissible functions, normalise this class with a three-point condition and show coercivity and weak lower semi-continuity of E_H on this normalised class with respect to $H^1(B;\mathbb{R}^3)$. We would then show that the surface X attaining the minimum is a relative minimiser for the functional E_H, and in particular with Lemma 2.2.2 we find that this solution X solves (2.31)–(2.33). For details we refer to Struwe's monograph [31], especially Theorem III.3.1.

2.2.3 Continuity

Before discussing the boundary behaviour of surfaces whose existence has so far been derived, we wish to point out a continuity property of H-surfaces.

Theorem 2.2.3. *Any weak solution X of* (2.31)–(2.33) *is continuous on $\bar{B}$.*

The theorem is a direct consequence of the following result.

Theorem 2.2.4. *Let $\varphi, \psi \in H^1(B; \mathbb{R}^3)$. Assume $Z \in H_0^1(B; \mathbb{R}^3)$ weakly solves*

$$\Delta Z = \varphi_u \wedge \psi_v + \psi_u \wedge \varphi_v \qquad \text{in } B.$$

Then $Z \in C^0(\bar{B}; \mathbb{R}^3)$.

For a proof we refer the reader to Theorem III.5.1 in Struwe [31]. Here we apply this theorem to derive Theorem 2.2.3 following [31].

Proof of Theorem 2.2.3. Write $X = X_0 + Z$ where $X_0 \in \mathcal{C}(\Gamma)$ (2.10) is a solution of Plateau's problem (2.7)–(2.9) and $Z \in H_0^1(B; \mathbb{R}^3)$. Then by definition of $\mathcal{C}(\Gamma)$ and by the maximum principle for harmonic functions [19] we infer $X_0 \in C^0(\bar{B}; \mathbb{R}^3)$. Furthermore, Z weakly solves

$$\Delta Z = 2H X_u \wedge X_v \qquad \text{in } B.$$

By Theorem 2.2.4 with $\varphi = \psi = X$ we have $Z \in C^0(\bar{B}; \mathbb{R}^3)$. Thus we conclude $X \in C^0(\bar{B}; \mathbb{R}^3)$. $\square$

Chapter 3

Boundary Regularity

In this chapter, we aim to discuss the boundary behaviour of minimal surfaces and of H-surfaces. For later reference, we introduce the Hölder spaces and their associated norm.

Let $\Omega \subset \mathbb{R}^n$ open. Consider $z : \Omega \to \mathbb{R}^n$, and let $0 < \beta \leq 1$. We first introduce the Hölder semi-norm

$$[z]_{C^{0,\beta}(\Omega;\mathbb{R}^n)} := \sup_{w_1, w_2 \in \Omega, w_1 \neq w_2} \frac{|z(w_1) - z(w_2)|}{|w_1 - w_2|^\beta}.$$

With this definition we arrive at the Hölder norm

$$\begin{aligned} \|z\|_{C^{0,\beta}(\Omega;\mathbb{R}^n)} &:= \|z\|_{C^0(\Omega;\mathbb{R}^n)} + [z]_{C^{0,\beta}(\Omega;\mathbb{R}^n)}, \\ \|z\|_{C^{m,\beta}(\Omega;\mathbb{R}^n)} &:= \|z\|_{C^m(\Omega;\mathbb{R}^n)} + \sum_{|\alpha|=m} [D^\alpha z]_{C^{0,\beta}(\Omega;\mathbb{R}^n)}. \end{aligned}$$

Then it is straightforward to define the Hölder spaces.

Definition 3.0.1 (Hölder spaces)**.** The *Hölder space* with exponent β is defined as

$$C^{m,\beta}(\Omega;\mathbb{R}^n) := \{z \in C^m(\bar{\Omega};\mathbb{R}^n) \mid \|z\|_{C^{m,\beta}(\Omega;\mathbb{R}^n)} < \infty\}.$$

Remark 3. i. The Hölder spaces $C^{0,\beta}(\Omega)$ are complete for $0 < \beta \leq 1$, see Satz 8.6.1 in [30].

ii. For $0 \leq \beta \leq \alpha \leq 1$ we have the embedding

$$C^{0,\alpha}(\Omega) \hookrightarrow C^{0,\beta}(\Omega),$$

where $C^{0,0}(\Omega) = C^0(\bar{\Omega})$, confer Beispiel 8.6.1 in [30].

Finally, we introduce a notion for the regularity of a closed Jordan curve $\Gamma \subset R^3$. Recall that in Chapter 2 we have introduced the closed Jordan curve $\Gamma \subset \mathbb{R}^3$ as an embedding of ∂B onto $\mathbb{R}^3$.

Definition 3.0.2. An open Jordan arc $\gamma \subset \mathbb{R}^3$ is said to be of class C^m (or of class $C^{m,\alpha}$, $0 < \alpha < 1$) for $m \geq 1$, if there is a homeomorphism $\tau : \mathbb{R}^3 \to \mathbb{R}^3$ such that

i. τ maps γ onto an interval $I := \{x = (0, 0, x^3) \in \mathbb{R}^3 | \; |x^3| < M\}$ for some constant $M > 0$,

ii. $\tau \in C^m(\mathbb{R}^3)$ (or $\tau \in C^{m,\alpha}(\mathbb{R}^3)$),

iii. the Jacobian $d\tau$ is non-singular on $\mathbb{R}^3$.

A closed Jordan curve Γ is said to be of class C^m (or of class $C^{m,\alpha}$) if Γ is a finite union of open subarcs of class C^m (or $C^{m,\alpha}$).

In case of a planar minimal surface, the three-point condition required for the solution of Plateau's problem is uniquely identified [22]. The surface X is of the form

$$X(u, v) = (X^1(u, v), X^2(u, v), 0),$$

where $X^1(u, v)$ and $X^2(u, v)$ are respectively the real and the imaginary part of an analytic function f, which maps the unit disc B conformally onto the interior of Γ. The regularity of the Riemannian mapping f depends on the regularity of the curve Γ, as has been outlined, amongst others, by Courant [3]. A first result on this relation has been made by Paul Painlevé, stating that if the boundary Γ is a regular curve of class C^{m+2}, then the mapping f is of class $C^m(\bar{B}; \mathbb{R}^3)$. An improved result by Oliver Dimon Kellogg [16] reads as follows: For a regular curve Γ of class $C^{m,\alpha}$, where $m \geq 1$ and $0 < \alpha < 1$, the mapping f is of class $C^{m,\alpha}(\bar{B}; \mathbb{R}^3)$.

In 1951, Hans Lewy demonstrated the first analogous results for minimal surfaces [17]. If the rectifiable curve Γ contains an open analytic regular arc γ, then the surface X in $\bar{B}$ can be generalised to a minimal surface beyond γ. This result has been used by Johannes Nitsche in [22] to develop the analogue of Kellogg's result for minimal surfaces.

Theorem 3.0.1. *If Γ is a regular Jordan curve of class $C^{m,\alpha}$ as defined in 3.0.2, where $m \geq 1$ and $0 < \alpha < 1$, then the surface X solving Plateau's problem (Definition 2.1.1) is of class $C^{m,\alpha}(\bar{B};\mathbb{R}^3)$.*

There have been various approaches to demonstrate this theorem. Stefan Hildebrandt has shown the statement for $m \geq 4$ [12, 15]. Erhard Heinz and Friedrich Tomi have developed methods to prove the statement in case that $m = 3$ and $\alpha = 0$ [11, 9]. Both their approaches rely on elliptic regularity theory for nonlinear partial differential equations. Finally, Nitsche approached the theorem from another point of view, and in this sense proves the statement in its full generality [21, 22]. Using analytic tools based on Lewy's result, Nitsche starts his considerations with the case $m = 1$, and develops thereon the cases $m > 1$. However, it has to be remarked that Nitsche's proof works for minimal surfaces, while the methods employed by Hildebrandt and by Heinz and Tomi allow for a generalisation to surfaces with constant mean curvature. We will exploit the reasoning of Heinz and Tomi to demonstrate Theorem 3.0.1 in the case of $m = 2$ for minimal surfaces. We will then apply these methods to the case of H-surfaces.

3.1 Heinz and Tomi on the Boundary Behaviour of Minimal Surfaces

Let $\Gamma \subset \mathbb{R}^3$ be a closed, rectifiable Jordan curve, and let $H \in \mathbb{R}$. In the previous chapter, we have established the existence of a surface $X : \bar{B} \to \mathbb{R}^3$ of class $C^2(B) \cap C^0(\bar{B})$ satisfying the elliptic system (2.31)–(2.32) with boundary conditions (2.33). In case that $H = 0$, these solutions are disc-type minimal surfaces; for non-vanishing mean curvature H, these solutions represent H-surfaces bounded by Γ. To establish the boundary regularity for surfaces X solving (2.31)–(2.33), we follow the approach of Heinz and Tomi in [11].

Let G be a bounded domain in the (u, v)-plane. We consider elliptic systems of the form

$$\Delta X = f(u, v, X, \nabla X), \tag{3.1}$$

where $f : G \times \mathbb{R}^3 \times \mathbb{R}^6 \to \mathbb{R}^3$ is a map with the property that for each $X \in H^1(G; \mathbb{R}^3)$ the function $f(u, v, X(u, v), \nabla X(u, v))$ is measurable as a function of (u, v). Moreover, we require f to satisfy

$$|f(u, v, X, p)| \leq \mu(\|X\|_{L^\infty}) p^2 \tag{3.2}$$

for all $(u, v) \in G$, $X, Y \in \mathbb{R}^3$, $p \in \mathbb{R}^6$, and where $\mu : \mathbb{R} \to \mathbb{R}$ is assumed to be monotonically increasing and continuous.

A weak solution X for (3.1) of class $H^1(G)$ satisfies $\|X\|_{L^\infty(G)} < \infty$ and

$$\int_G \big(\nabla X \nabla Z + f(u, v, X, \nabla X) Z\big)\, dw = 0 \quad \text{for all} \;\; Z \in C_c^\infty(G). \tag{3.3}$$

3.1.1 A Regularity Result

In this section, we will prove a regularity result for weak solutions of (3.1), where we follow the considerations of Tomi [32] and Struwe [29]. The result reads

Theorem 3.1.1. *Let $X \in H^1(G; \mathbb{R}^3)$ weakly solve (3.1), where the right-hand side f is measurable as a function of (u, v) and satisfies (3.2) with a monotonically increasing, continuous μ. Assume that there exists some $a \in \mathbb{R}^3$ with*

$$\|X - a\|_{L^\infty(G;\mathbb{R}^3)} \mu(\|X\|_{L^\infty(G;\mathbb{R}^3)}) < \frac{1}{2}. \tag{3.4}$$

Then $X \in C^{1,\beta}(G; \mathbb{R}^3)$ for all β with $0 < \beta < 1$.

The proof relies on two results. The first lemma is due to Tomi [32].

Lemma 3.1.2. *Let $X \in H^1(G; \mathbb{R}^3)$ weakly solve (3.1), where the right-hand side f is measurable as a function of (u, v) and satisfies (3.2) with a monotonically increasing, continuous μ. Assume X satisfies (3.4). Then $X \in C^\beta(G; \mathbb{R}^3)$ for all β with $0 < \beta < \frac{1}{2}$.*

Proof. Let $w_0 \in G$ and let $r > 0$ be such that $\bar{B}_r(w_0) \subset G$. Let Y be the weakly harmonic function in $\bar{B}_r(w_0)$ with $Y|_{\partial B_r(w_0)} = X$ in a trace sense. We define

$$Z(w) := \begin{cases} X(w) - Y(w) & \text{for } w \in B_r(w_0), \\ 0 & \text{else.} \end{cases}$$

We note that Z satisfies (3.1) weakly in $B_r(w_0)$ and has compact support in G. In particular, Z is an admissible test function for (3.3). Thus from (3.2) and (3.3) we obtain

$$\int\limits_{B_r(w_0)} |\nabla X|^2 \, dw \leq \int\limits_{B_r(w_0)} |\nabla X||\nabla Y| + |\nabla X|^2 |Z| \mu(\|X\|_{L^\infty(G;\mathbb{R}^3)}) \, dw. \tag{3.5}$$

Using the maximum principle for harmonic functions (p.72 in [26]) we obtain $\|Z\|_{L^\infty} \leq 2\|X - a\|_{L^\infty}$, where a is such that X satisfies (3.4). Thus, (3.4), (3.5) and Young's inequality yield

$$\begin{aligned} \big(1 - 2\|X - a\|_{L^\infty} \mu(\|X\|_{L^\infty})\big) & \int\limits_{B_r(w_0)} |\nabla X|^2 \, dw \\ & \leq \frac{\varepsilon}{2} \int\limits_{B_r(w_0)} |\nabla X|^2 \, dw + \frac{1}{2\varepsilon} \int\limits_{B_r(w_0)} |\nabla Y|^2 \, dw, \end{aligned}$$

where $\varepsilon > 0$. We choose $\varepsilon = \big(1 - 2\|X - a\|_{L^\infty(G;\mathbb{R}^3)} \mu(\|X\|_{L^\infty(G;\mathbb{R}^3)})\big)$. Then

$$\int\limits_{B_r(w_0)} |\nabla X|^2 \, dw \leq \kappa \int\limits_{B_r(w_0)} |\nabla Y|^2 \, dw,$$

where $\kappa = \big(1 - 2\|X - a\|_{L^\infty(G;\mathbb{R}^3)} \mu(\|X\|_{L^\infty(G;\mathbb{R}^3)})\big)^{-2}$. But then Theorem 1.10.2 and Theorem 3.5.2 in Morrey [19] give that $X \in C^\beta(G;\mathbb{R}^3)$, where $\beta = \frac{1}{2\kappa}$.

We can apply the same argument for $G = B_R(\eta)$ for some sufficiently small $R > 0$ and some $\eta \in G$. With $a = X(\eta)$ we then see that $\|X - a\|_{L^\infty(G;\mathbb{R}^3)}$ is arbitrarily close to zero, so that $\kappa = 1 + \delta$ is attained for any $\delta > 0$. Thus $X \in C^\beta(G;\mathbb{R}^3)$ for any $0 < \beta < \frac{1}{2}$. □

Based on Lemma 3 in Tomi [32], we state a result which allows us to represent X in terms of a harmonic function in a ball and the Green's function for the ball.

Lemma 3.1.3. *Let $w_0 \in G$. Choose $R > 0$ such that $\bar{B}_R(w_0) \subset G$. Let Y be a harmonic function in $B_R(w_0)$ coinciding with X on $\partial B_R(w_0)$, and let $G = G(w,z)$ be the harmonic Green's function for $B_R(w_0)$. For $w = (u,v) \in B_R(w_0)$ we then have the representations*

$$X(w) = Y(w) + \int_{B_R(w_0)} G(w,z) f\big(z, X(z), \nabla X(z)\big)\, dz \tag{3.6}$$

and

$$\begin{aligned} \frac{\partial X}{\partial u}(w) &= \frac{\partial Y}{\partial u}(w) + \int_{B_R(w_0)} \frac{\partial G}{\partial u}(w,z) f\big(z, X(z), \nabla X(z)\big)\, dz, \\ \frac{\partial X}{\partial v}(w) &= \frac{\partial Y}{\partial v}(w) + \int_{B_R(w_0)} \frac{\partial G}{\partial v}(w,z) f\big(z, X(z), \nabla X(z)\big)\, dz. \end{aligned} \tag{3.7}$$

For the proof we refer the reader to Theorem 1.17 in Vekua [33]. For the third result we follow Struwe's reasoning in [29].

Theorem 3.1.4. *Let $X \in H^1 \cap L^\infty(G;\mathbb{R}^3)$ weakly solve (3.1), where the right-hand side f is measurable as a function of (u,v) and satisfies (3.2) with a monotonically increasing, continuous μ. Suppose $X \in C^\alpha$ for some $\alpha > 0$ and assume X satisfies (3.4). Then $X \in C^{1,\beta}$ for any $\beta < 1$.*

Proof. Let $w_0 \in G$ and let $R > 0$. We split $X = Y + Z$ on $B_R(w_0)$, where $\Delta Y = 0$ and $Z \in H^1_0(B_R(w_0);\mathbb{R}^3)$. We note that $X|_{\partial B_R(w_0)} = Y|_{\partial B_R(w_0)}$ since $Z \in H^1_0(B_R(w_0);\mathbb{R}^3)$. With the maximum principle [26] we infer that

$$\begin{aligned} \sup_{\zeta,\eta \in B_R(w_0)} |Y(\zeta) - Y(\eta)| &\le \sup_{\substack{\zeta \in \partial B_R(w_0) \\ \eta \in B_R(w_0)}} |Y(\zeta) - Y(\eta)| \\ &\le \sup_{\zeta,\eta \in \partial B_R(w_0)} |Y(\zeta) - Y(\eta)| \\ &\le \sup_{\zeta,\eta \in B_R(w_0)} |X(\zeta) - X(\eta)| \\ &\le CR^\alpha, \end{aligned}$$

where $C = [X]_{C^{0,\alpha}}$. Furthermore by choice of Y and Z there holds $Z = X - Y$, so that we have

$$\sup_{\zeta,\eta\in B_R(w_0)} |Z(\zeta)-Z(\eta)| \le 2 \sup_{\zeta,\eta\in B_R(w_0)} |X(\zeta)-X(\eta)| \le 2CR^\alpha. \tag{3.8}$$

Using the splitting $X = Y + Z$ and Campanato's result (Satz 10.2.2 in [30]) we estimate for $0 < r < R \le 1$

$$\begin{aligned}\int_{B_r(w_0)} |\nabla X|^2\,dw &\le 2\int_{B_r(w_0)} |\nabla Y|^2\,dw + 2\int_{B_r(w_0)} |\nabla Z|^2\,dw \\ &\le C\Big(\frac{r}{R}\Big)^3 \int_{B_R(w_0)} |\nabla Y|^2\,dw + 2\int_{B_r(w_0)} |\nabla Z|^2\,dw \\ &\le C\Big(\frac{r}{R}\Big)^3 \int_{B_R(w_0)} |\nabla X|^2\,dw + C\int_{B_R(w_0)} |\nabla Z|^2\,dw.\end{aligned}$$

Moreover, by (3.1), (3.2), (3.4) and (3.8) we find

$$\begin{aligned}\int_{B_R(w_0)} |\nabla Z|^2\,dw &= \int_{B_R(w_0)} f(u,v,X,\nabla X)Z\,dw \\ &\le C\int_{B_R(w_0)} |\nabla X|^2 dw \sup_{B_R(w_0)} |Z| \\ &\le CR^\alpha,\end{aligned} \tag{3.9}$$

since $X \in H^1(G;\mathbb{R}^3)$. Thus for the non-decreasing function

$$\Phi(r) := \int_{B_r(w_0)} |\nabla X|^2\,dw, \qquad 0 < r \le 1,$$

there holds

$$\Phi(r) \le C\Big(\frac{r}{R}\Big)^3 \Phi(R) + CR^\alpha, \qquad 0 < r < R \le 1. \tag{3.10}$$

Campanato's useful Lemma (Lemma 10.3.2 in [30]) yields

$$\Phi(r) \le Cr^\alpha, \qquad 0 < r \le 1.$$

Inserting this bound into (3.9) gives for $0 < r < R \le 1$

$$\int_{B_R(w_0)} |\nabla Z|^2 \le Cr^\alpha R^\alpha \le CR^{2\alpha}.$$

Thus we obtain the improved bound

$$\Phi(r) \leq C\big(\frac{r}{R}\big)^3 \Phi(R) + CR^{2\alpha}, \qquad 0 < r < R \leq 1.$$

By iterating we then arrive at

$$\Phi(r) \leq C r^{\beta} \tag{3.11}$$

for any $\beta < 3$.

Campanato's estimate, Poincaré's inequality (Satz 10.2.2 and Satz 8.6.6 in [30]), the maximum principle and monotonicity of the integral also imply that for any $0 < r < R \leq 1$ we find

$$\begin{aligned}
&\int\limits_{B_r(w_0)} |\nabla X - (\overline{\nabla X})_r|^2 \, dw \\
&\qquad \leq 2 \int\limits_{B_r(w_0)} |\nabla Y - (\overline{\nabla Y})_r|^2 \, dw + 2 \int\limits_{B_r(w_0)} |\nabla Z - (\overline{\nabla Z})_r|^2 \, dw \\
&\qquad \leq C\big(\frac{r}{R}\big)^5 \int\limits_{B_R(w_0)} |\nabla Y - (\overline{\nabla Y})_R|^2 \, dw + C \int\limits_{B_r(w_0)} |\nabla Z|^2 \, dw \\
&\qquad \leq C\big(\frac{r}{R}\big)^5 \int\limits_{B_R(w_0)} |\nabla X - (\overline{\nabla X})_R|^2 \, dw + C \int\limits_{B_R(w_0)} |\nabla Z|^2 \, dw,
\end{aligned}$$

where $(\overline{\nabla X})_r = \fint_{B_r(0)} \nabla X(w) \, dw$. Similarly to (3.9) we have in view of (3.11)

$$\int_{B_R(w_0)} |\nabla Z|^2 \, dw \leq C \int_{B_R(w_0)} |\nabla X|^2 \, dw \sup_{B_R(w_0)} |Z| \leq C R^{\alpha+\beta}$$

for any $\beta < 3$. Thus the function $\Psi : (0,1] \to \mathbb{R}$ defined as

$$\Psi(r) := \int_{B_r(w_0)} |\nabla X - \big(\overline{\nabla X}\big)_r|^2 \, dw$$

satisfies

$$\Psi(r) \leq C\big(\frac{r}{R}\big)^5 \Psi(R) + CR^{\alpha+\beta}, \qquad 0 < r < R \leq 1.$$

Moreover there holds that

$$\Psi(r) \le \int_{B_r(w_0)} |\nabla X - (\overline{\nabla X})_R|^2 \, dw \le \Psi(R).$$

Thus Ψ is non-decreasing. With Campanato's useful Lemma (Lemma 10.3.2 [30]) we obtain

$$\Psi(r) \le C r^{\alpha+\beta}, \qquad 0 < r \le 1. \tag{3.12}$$

We can then choose $\beta < 3$ so that $\alpha + \beta = 3 + 2\delta > 3$ for some $\delta > 0$. Then by Campanato's embedding theorem (Satz 8.6.5 in [30]) we have in view of (3.12)

$$\nabla X \in \mathcal{L}^{2,3+2\delta} \hookrightarrow C^{\delta},$$

where

$$\mathcal{L}^{2,3+2\delta}(G;\mathbb{R}^3) := \{X \in L^2(G;\mathbb{R}^3) | \; [X]_{\mathcal{L}^{2,3+2\delta}(G;\mathbb{R}^3)} < \infty\}$$

defines the Campanato space with semi-norm

$$[X]^2_{\mathcal{L}^{2,3+2\delta}(G;\mathbb{R}^3)} := \sup_{\substack{w_0 \in G \\ 0<r<\min\{1, \text{diam } G\}}} r^{-(3+2\delta)} \int_{B_r(w_0)\cap G} |X - X_{w_0,r}|^2 \, dw$$

for $X_{w_0,r} := \fint_{B_r(w_0)\cap G} X(w) dw$, see Definition 8.6.3 in [30]. Equation (3.7) then implies $f(u, v, X, \nabla X) \in C^{\delta}$. From (3.1) we obtain $\Delta X \in C^{\delta}$. With Schauder's theory (Satz 10.5.1 in [30]) we infer

$$X \in C^{2,\delta} \hookrightarrow \bigcap_{0<\beta<1} C^{1,\beta}.$$

$\square$

With these statements at hand, we directly obtain Theorem 3.1.1.

3.1.2 First Main Result

We proceed with Heinz and Tomi [11]. We introduce polar coordinates r, φ with $u + iv = re^{i\varphi}$. We set

$$D_r := \frac{\partial}{\partial r}, \qquad D_\varphi := \frac{\partial}{\partial \varphi}.$$

Furthermore, we consider the domain

$$S_{R,\Theta} = \{re^{i\varphi} \big|\ R < r < 1,\ |\varphi| < \Theta\}$$

for $0 < R < 1$, and $0 < \Theta \leq \pi$. We then have

Theorem 3.1.5. *Let $X \in H^1(S_{R,\Theta};\mathbb{R}^3) \cap C^2(S_{R,\Theta};\mathbb{R}^3) \cap C^0(\bar{S}_{R,\Theta};\mathbb{R}^3)$ be a solution of*

$$|\Delta X| \leq \alpha |\nabla X|^2 \tag{3.13}$$

for some constant $\alpha > 0$. Moreover, we assume the following boundary conditions

$$X_k(e^{i\varphi}) = 0 \quad \text{for } |\varphi| \leq \Theta,\ k \in \{1,2\}, \tag{3.14}$$

and

$$\lim_{r\to 1} \int_{-\tilde{\Theta}}^{\tilde{\Theta}} |D_r X_3|\, d\varphi = 0, \tag{3.15}$$

for all $\tilde{\Theta}$ with $0 < \tilde{\Theta} < \Theta$. Then there holds

$$X \in \bigcap_{0<\beta<1} C^{1,\beta}(\bar{S}_{\tilde{R},\tilde{\Theta}})$$

for all $\tilde{R}, \tilde{\Theta}$ with $R < \tilde{R} < 1$ and $0 < \tilde{\Theta} < \Theta$.

Proof. Let $G := \{re^{i\varphi} \big|\ R < r < R^{-1},\ |\varphi| < \Theta\}$. Define each component of a vector-valued function $Y : G \to \mathbb{R}^3$ as

$$Y_k(re^{i\varphi}) := \begin{cases} X_k(re^{i\varphi}) & \text{if } r \leq 1, \\ (2\delta_{k,3} - 1) X_k(\frac{1}{r}e^{i\varphi}) & \text{if } r > 1, \end{cases}$$

for $k \in \{1,2,3\}$, where $\delta_{k,3}$ denotes the Kronecker delta. Define further $a : G \to \mathbb{R}^3$ as

$$a := \begin{cases} |\nabla Y|^{-2} \Delta Y & \text{if } \nabla Y \neq 0 \text{ and } r \neq 1, \\ 0 & \text{else.} \end{cases}$$

Then a is a measurable vector-valued function and $|a| \leq \alpha$ due to (3.13). Moreover by definition of a, we have that Y satisfies

$$\Delta Y = |\nabla Y|^2 a \tag{3.16}$$

for $r \neq 1$. We claim that Y is a weak solution of (3.16) in all of G. Indeed, Y is continuous in G, and since $X \in H^1(S_{R,\Theta};\mathbb{R}^3)$ we have $\int_{r\neq 1} |\nabla Y|^2 \, dw < \infty$. Thus $Y \in H^1(G;\mathbb{R}^3)$. We now derive the weak formulation satisfied by Y. Let $Z \in C_c^\infty(G;\mathbb{R}^3)$. We choose $s \in \mathbb{R}$ such that $R < s < 1$. Then we obtain

$$
\begin{aligned}
\int_{R<r<s} &\big(\nabla Y \nabla Z + |\nabla Y|^2 a Z\big)\, dw + \int_{\frac{1}{s}<r<\frac{1}{R}} \big(\nabla Y \nabla Z + |\nabla Y|^2 a Z\big)\, dw \\
&= \int_{R<r<s} \big(-\Delta Y + |\nabla Y|^2 a\big) Z \, dw + \int_{\frac{1}{s}<r<\frac{1}{R}} \big(-\Delta Y + |\nabla Y|^2 a\big) Z \, dw \\
&\quad + \Big[\int_{-\tilde\Theta}^{\tilde\Theta} \big(D_r Y\big) Z r \, d\varphi\Big]_{r=R}^{s} + \Big[\int_{-\tilde\Theta}^{\tilde\Theta} \big(D_r Y\big) Z r \, d\varphi\Big]_{r=\frac{1}{s}}^{\frac{1}{R}} \\
&= \Big[\int_{-\tilde\Theta}^{\tilde\Theta} \big(D_r Y\big) Z r \, d\varphi\Big]_{r=\frac{1}{s}}^{s} \\
&= \sum_{k=1}^{3} \Big[\int_{-\tilde\Theta}^{\tilde\Theta} \big(D_r Y_k\big) Z_k r \, d\varphi\Big]_{r=\frac{1}{s}}^{s},
\end{aligned}
$$

for some suitable $\tilde\Theta$ with $0 < \tilde\Theta < \Theta$. For the first equality we integrated by parts, for the second we used that Y solves (3.16) where $r \neq 1$ and that Z has compact support in G.

Note that for $r > 1$ we have

$$
D_r Y_k\big(re^{i\varphi}\big) = (1 - 2\delta_{k,3}) \frac{1}{r^2} D_r X_k\big(\frac{1}{r} e^{i\varphi}\big), \quad k \in \{1,2,3\},
$$

so that

$$
\begin{aligned}
\sum_{k=1}^{2} &\Big|\Big[\int_{-\tilde\Theta}^{\tilde\Theta} \big(D_r Y_k\big) Z_k r \, d\varphi\Big]_{r=\frac{1}{s}}^{s}\Big| \\
&\leq \sum_{k=1}^{2} \int_{-\tilde\Theta}^{\tilde\Theta} |D_r X_k(se^{i\varphi})| |Z_k(se^{i\varphi}) - Z_k(\frac{1}{s} e^{i\varphi})| s \, d\varphi \\
&\leq C\big(\frac{1}{s} - s\big)\Big(\int_{-\tilde\Theta}^{\tilde\Theta} |D_r X(se^{i\varphi})| s \, d\varphi\Big),
\end{aligned}
$$

where we used the mean value theorem [25]. Furthermore we have

$$\Big|\Big[\int_{-\tilde{\Theta}}^{\tilde{\Theta}} \big(D_r Y_3\big) Z_3 r \, d\varphi\Big]_{r=\frac{1}{s}}^{s}\Big| = \int_{-\tilde{\Theta}}^{\tilde{\Theta}} |D_r X_3(se^{i\varphi})||Z_3(se^{i\varphi}) + Z_3(\frac{1}{s}e^{i\varphi})| s \, d\varphi$$

$$\leq Cs \int_{-\tilde{\Theta}}^{\tilde{\Theta}} |D_r X_3(se^{i\varphi})| \, d\varphi \ \to 0, \qquad (s \to 1),$$

where the limit is obtained using (3.15). Moreover, as $X \in H^1(S_{R,\Theta}; \mathbb{R}^3)$ and as $S_{R,\Theta}$ is a bounded domain, we have with Hölder's inequality

$$\Big| \int_{S_{R,\Theta}} |\nabla X| \, dw \Big| \leq C\big(\int_{S_{R,\Theta}} |\nabla X|^2 \, dw \big)^{\frac{1}{2}} < \infty,$$

so that for a sequence $(s_\nu)_{\nu \in \mathbb{N}}$ with $s_\nu \to 1$ as $(\nu \to \infty)$ we have

$$(\frac{1}{s_\nu} - s_\nu)\Big(\int_{-\tilde{\Theta}}^{\tilde{\Theta}} |D_r X(s_\nu e^{i\varphi})| s_\nu \, d\varphi \Big) \to 0 \qquad (\nu \to \infty).$$

We conclude that by possibly passing to a subsequence, we have

$$\begin{aligned}
&\int_G \big(\nabla Y \nabla Z + |\nabla Y|^2 a Z\big) \, dw \\
&\quad = \lim_{s \to 1} \Big(\int_{R<r<s} \big(\nabla Y \nabla Z + |\nabla Y|^2 a Z\big) \, dw + \int_{\frac{1}{s}<r<\frac{1}{R}} \big(\nabla Y \nabla Z + |\nabla Y|^2 a Z\big) \, dw \Big) \\
&\quad = \lim_{s \to 1} \Big(\sum_{k=1}^{3} \Big[\int_{-\tilde{\Theta}}^{\tilde{\Theta}} \big(D_r Y_k\big) Z_k r \, d\varphi \Big]_{r=\frac{1}{s}}^{s} \Big) \\
&\quad = 0,
\end{aligned}$$

that is, Y weakly solves (3.16) in G.

Now we can invoke Theorem 3.1.1 to conclude. Indeed, Y solves (3.1) with $f(u, v, Y, \nabla Y) := a|\nabla Y|^2$. Thus (3.2) is satisfied for $\mu(s) := a$ which is continuous and monotonically increasing as a constant function and uniformly bounded by α by (3.13). Moreover, $Y \in C^0(\bar{G})$ since by assumption X is continuous on $\bar{S}_{R,\Theta}$. We can now cover $\bar{G}$ with sufficiently small balls $B_\varepsilon(\eta)$ for $\eta \in G$ and some $\varepsilon > 0$. On each ball choose $b := Y(\eta)$. Then there holds $\|Y - b\|_{L^\infty(G;\mathbb{R}^3)} < \frac{1}{2a}$ by continuity of Y, i.e. (3.4) is satisfied. We therefore conclude $Y \in \bigcap_{0<\beta<1} C^{1,\beta}(G; \mathbb{R}^3)$, and hence $X \in \bigcap_{0<\beta<1} C^{1,\beta}(\bar{S}_{\tilde{R},\tilde{\Theta}};\mathbb{R}^3)$ for all $\tilde{R},\tilde{\Theta}$ with $R<\tilde{R}<1$ and $0<\tilde{\Theta}<\Theta$ as required.

□

Remark 4. Let us emphasise that the bound in (3.4) required for Theorem 3.1.1 can easily be obtained for minimal surfaces and H-surfaces. Indeed, the maximum principle for harmonic functions directly implies a C^0-bound on the solution vector in terms of the boundary values in the case of minimal surfaces. Similarly, for H-surfaces such a bound can be obtained by the maximum principle for sub-harmonic functions, see Theorem III.3.1 in Struwe [31]. To see this, note that the admissible class of H-surfaces is

$$\mathcal{C}_H(\Gamma) := \left\{ X \in \mathcal{C}(\Gamma) |\ \|X\|_{L^\infty} \le \frac{1}{|H|} \right\},$$

where H has to satisfy $|H|R \le 1$ for $\Gamma \subset B_R(0) \subset \mathbb{R}^3$. These H-surfaces then satisfy $\|X\|_{L^\infty} \le R$ [31]. Considering the case $R=1$ we note that choosing $f = 2HX_u \wedge X_v$ in (3.1) implies that $\mu \equiv H$ in order for f to comply with (3.2). Hence (3.4) is fulfilled since

$$\|X\|_{L^\infty}\mu(\|X\|_{L^\infty}) \le RH < \frac{1}{2}.$$

if we assume $H < \frac{1}{2}$, see Section 3.3.

3.1.3 Second Main Result

We proceed with Heinz and Tomi's discussion in [11]. We consider a conformally parametrised surface $X = X(u,v)$ whose boundary is given by a regular curve of class C^m for $m \ge 2$. Concretely, let $X : \bar{B} \to \mathbb{R}^3$ be a vector-valued function of class $C^1(B;\mathbb{R}^3) \cap C^0(\bar{B};\mathbb{R}^3)$ satisfying

$$r|D_r X| = |D_\varphi X|, \tag{3.17}$$

$$D_r X \cdot D_\varphi X = 0, \tag{3.18}$$

with boundary conditions

$$X_k(e^{i\varphi}) = g_k\big(X_3(e^{i\varphi})\big) \qquad \text{for } |\varphi| \le \Theta,\ k \in \{1,2\}, \tag{3.19}$$

$$\big(X_3(e^{i\varphi}) - X_3(e^{i\psi})\big)(\varphi - \psi) \ge 0 \qquad \text{for } |\varphi| \le \Theta,\ |\psi| \le \Theta, \tag{3.20}$$

where $\Theta \in \mathbb{R}$ with $0 < \Theta \le \pi$. The functions $g_k : (-\delta, \delta) \to \mathbb{R}$ for $k \in \{1,2\}$ are of class $C^m\big((-\delta,\delta)\big)$, where δ is chosen such that $\{X_3(e^{i\varphi}) \mid |\varphi| \le \Theta\} \subset (-\delta, \delta)$. With no loss of generality, we set

$$g_k(0) = g_k'(0) = 0 \qquad k \in \{1,2\}. \tag{3.21}$$

We set $M(\Theta) := \{X(e^{i\varphi}) \mid |\varphi| \le \Theta\}$ and we consider a transformation V defined in a neighbourhood of $M(\Theta)$ such that $Y = V \circ X$ satisfies for $k \in \{1,2\}$

$$Y_k := X_k - g_k(X_3), \tag{3.22}$$

$$Y_3 := X_3 + h(X_3)^{-1}\Big(g_1'(X_3)\big(X_1 - g_1(X_3)\big) + g_2'(X_3)\big(X_2 - g_2(X_3)\big)\Big), \tag{3.23}$$

where

$$h(X_3) := 1 + g_1'(X_3)^2 + g_2'(X_3)^2.$$

Note that we choose $\Theta > 0$ sufficiently small, so that V is bijective in a neighbourhood of $M(\Theta)$ with a nowhere vanishing Jacobian. Then Y satisfies

$$Y_k(e^{i\varphi}) = 0 \qquad \text{for } |\varphi| \le \Theta,\ k \in \{1,2\}.$$

Lemma 3.1.6. *Let $X \in C^1(B;\mathbb{R}^3) \cap C^0(\bar{B};\mathbb{R}^3)$ satisfy* (3.17)*,* (3.18)*,* (3.19)*. Consider the transformation V defined as above in* (3.22)*–*(3.23) *on a neighbourhood Ω of $M(\Theta)$. Choose $R < 1$ such that $X(\bar{S}_{R,\Theta}) \subset \Omega$. Then there holds for $Y : S_{R,\Theta} \to \mathbb{R}^3$ with $Y(re^{i\varphi}) := VX(re^{i\varphi})$ in $S_{R,\Theta}$*

$$|D_r Y_3| \le C\frac{1}{r}\Big(|D_\varphi X|^{\frac{3}{4}}\big(|D_\varphi Y| - |D_\varphi Y_3|\big)^{\frac{1}{4}} + |D_\varphi X|\big(|Y_1| + |Y_2|\big)\Big) \tag{3.24}$$

Proof. We have with (3.22), (3.23) and (3.18)

$$
\begin{aligned}
0 &= -\sum_{k=1}^{2} g_k'(X_3) D_r X_k D_\varphi X_3 + \sum_{k=1}^{2} g_k'(X_3) D_r X_k D_\varphi X_3 \\
&= \sum_{k=1}^{2} \big(D_\varphi X_k - g_k'(X_3) D_\varphi X_3\big) D_r X_k + \sum_{k=1}^{2} g_k'(X_3)\big(D_r X_k - g_k'(X_3) D_r X_3\big) D_\varphi X_3 \\
&= \sum_{k=1}^{2} D_\varphi Y_k D_r X_k + \sum_{k=1}^{2} g_k'(X_3) D_r Y_k D_\varphi X_3 \\
&= \sum_{k=1}^{2} D_\varphi Y_k D_r X_k + \underbrace{\Big(\sum_{k=1}^{2} g_k'(X_3) D_r Y_k + h(X_3) D_r X_3\Big)}_{=:\Phi} D_\varphi X_3 \\
&= \sum_{k=1}^{2} D_\varphi Y_k D_r X_k + \Phi D_\varphi X_3.
\end{aligned}
$$

Therefrom we obtain with Cauchy-Schwarz's inequality and with (3.17)

$$
\begin{aligned}
|\Phi||D_\varphi X_3| &= \Big|\sum_{k=1}^{2} D_\varphi Y_k D_r X_k\Big| \\
&\leq \Big(\sum_{k=1}^{2} (D_\varphi Y_k)^2\Big)^{\frac{1}{2}} |D_r X| \qquad (3.25) \\
&= \Big(\sum_{k=1}^{2} (D_\varphi Y_k)^2\Big)^{\frac{1}{2}} \frac{1}{r} |D_\varphi X|.
\end{aligned}
$$

Moreover, the definition of Φ yields with (3.22), (3.23) and (3.17)

$$
\begin{aligned}
|\Phi| &= \big|\sum_{k=1}^{2} g_k'(X_3)D_rY_k + h(X_3)D_rX_3\big| \\
&= \big|\sum_{k=1}^{2} g_k'(X_3)\big(D_rX_k - g_k'(X_3)D_rX_3\big) + \big(1+\sum_{k=1}^{2} g_k'(X_3)^2\big)D_rX_3\big| \\
&= \big|\sum_{k=1}^{2} g_k'(X_3)D_rX_k + D_rX_3\big| \\
&\leq \big|\sum_{k=1}^{2} g_k'(X_3)|D_rX_k|\big| + |D_rX_3| \\
&= \frac{1}{r}\big|\sum_{k=1}^{2} g_k'(X_3)|D_\varphi X_k|\big| + \ \frac{1}{r}|D_\varphi X_3| \\
&= \frac{1}{r}\big|\sum_{k=1}^{2} g_k'(X_3)|(D_\varphi Y_k + g_k'(X_3)D_\varphi X_3)|\big| + \ \frac{1}{r}|D_\varphi X_3| \\
&\leq \frac{1}{r}\big|\sum_{k=1}^{2} g_k'(X_3)|D_\varphi Y_k|\big| + \frac{1}{r}\big|\sum_{k=1}^{2} |g_k'(X_3)^2 D_\varphi X_3|\big| + \ \frac{1}{r}|D_\varphi X_3| \\
&\leq C_0\frac{1}{r}\big(\sum_{k=1}^{2}(D_\varphi Y_k)^2\big)^{\frac{1}{2}} + C_1\frac{1}{r}|D_\varphi X_3| \\
&\leq C_2\frac{1}{r}\Big(\big(\sum_{k=1}^{2}(D_\varphi Y_k)^2\big)^{\frac{1}{2}} + |D_\varphi X_3|\Big),
\end{aligned}
$$

with positive constants C_0, C_1, C_2. From this estimate we obtain with (3.25), Young's inequality and (3.22), (3.23)

$$
\begin{aligned}
\Phi^2 &\leq \frac{C_2}{r}\Big(|\Phi|\big(\sum_{k=1}^{2}(D_\varphi Y_k)^2\big)^{\frac{1}{2}} + |\Phi||D_\varphi X_3|\Big) \\
&\leq \frac{1}{2}\Phi^2 + \frac{1}{2}\frac{C_2^2}{r^2}\sum_{k=1}^{2}(D_\varphi Y_k)^2 + \frac{C_2}{r^2}\big(\sum_{k=1}^{2}(D_\varphi Y_k)^2\big)^{\frac{1}{2}}|D_\varphi X| \\
&\leq \frac{1}{2}\Phi^2 + \frac{1}{2}\frac{C_3^2}{r^2}\big(\sum_{k=1}^{2}(D_\varphi Y_k)^2\big)^{\frac{1}{2}}|D_\varphi X|,
\end{aligned}
$$

that is

$$|\Phi| \le \frac{C_3}{r}\big(\sum_{k=1}^{2}(D_\varphi Y_k)^2\big)^{\frac{1}{4}}|D_\varphi X|^{\frac{1}{2}}.$$

Combining this estimate with the following

$$\begin{aligned}\sum_{k=1}^{2}(D_\varphi Y_k)^2 &= |D_\varphi Y|^2 - |D_\varphi Y_3|^2\\ &= (|D_\varphi Y| + |D_\varphi Y_3|)(|D_\varphi Y| - |D_\varphi Y_3|)\\ &\le C|D_\varphi X|(|D_\varphi Y| - |D_\varphi Y_3|),\end{aligned}$$

we obtain

$$|\Phi| \le C\frac{1}{r}|D_\varphi X|^{\frac{3}{4}}(|D_\varphi Y| - |D_\varphi Y_3|)^{\frac{1}{4}}.$$

With (3.22) and (3.23) we have

$$D_r Y_3 = h(X_3)^{-1}\Phi + \sum_{k=1}^{2}\frac{d}{dX_3}\Big(\frac{g'(X_3)}{h(X_3)}\Big)(D_r X_3)Y_k,$$

so that we conclude

$$\begin{aligned}|D_r Y_3| &\le C|\Phi| + C|D_r X_3|\Big|\sum_{k=1}^{2} y_k\Big|\\ &\le C\frac{1}{r}|D_\varphi X|^{\frac{3}{4}}(|D_\varphi Y| - |D_\varphi Y_3|)^{\frac{1}{4}} + C|D_r X|\sum_{k=1}^{2}|y_k|\\ &\le C\frac{1}{r}\Big(|D_\varphi X|^{\frac{3}{4}}(|D_\varphi Y| - |D_\varphi Y_3|)^{\frac{1}{4}} + |D_\varphi X|\sum_{k=1}^{2}|y_k|\Big).\end{aligned}$$

□

The next lemma states a property on the behaviour of arc lengths after a nonlinear transformation.

Lemma 3.1.7. *Let* $X \in C^1(B;\mathbb{R}^3) \cap C^0(\bar{B};\mathbb{R}^3)$ *with*

$$\sup_{0\le r\le 1}\int_{-\Theta}^{\Theta}|D_\varphi X(re^{i\varphi})|\,d\varphi < \infty$$

for some Θ with $0 < \Theta \leq \pi$. Let Ω be a neighbourhood of $X(\bar{S}_{R,\Theta})$ for $0 < R < 1$. Consider a continuously differentiable map $U : \Omega \to \mathbb{R}^3$. Let $J_U(z)$ denote the Jacobian of U in z. Assume for all $z \in \{X(e^{i\varphi})\big|\ |\varphi| \leq \Theta\}$ there holds

$$\limsup_{r\to 1} \int_{-\Theta}^{\Theta} |D_\varphi J_U(z) X(re^{i\varphi})|\, d\varphi = \int_{-\Theta}^{\Theta} |D_\varphi J_U(z) X(e^{i\varphi})|\, d\varphi. \tag{3.26}$$

Then for $Y(re^{i\varphi}) := UX(re^{i\varphi}),\ R \leq r \leq 1, |\varphi| \leq \Theta$ there holds

$$\lim_{r\to 1} \int_{-\Theta}^{\Theta} |D_\varphi Y(re^{i\varphi})|\, d\varphi = \int_{-\Theta}^{\Theta} |D_\varphi Y(e^{i\varphi})|\, d\varphi < \infty. \tag{3.27}$$

Proof. Finiteness of the right-hand side in (3.27) follows by continuous differentiability of U and X.

For given $\varepsilon > 0$ we can find $\delta(\varepsilon)$ with $0 < \delta(\varepsilon) < 1 - R$ and a partition of the interval $[-\Theta, \Theta]$

$$-\Theta = \varphi_0 < \varphi_1 < ... < \varphi_l < \varphi_{l+1} = \Theta,$$

where $\varphi_{k+1} - \varphi_k < \delta(\varepsilon)$ so that there holds

$$\varphi_k \leq \varphi \leq \varphi_{k+1}, \qquad 1 - \delta(\varepsilon) \leq r \leq 1,$$

and

$$|D_\varphi Y(re^{i\varphi}) - J_V(X(e^{i\varphi_k})) D_\varphi X(re^{i\varphi})| \leq \varepsilon |D_\varphi X(re^{i\varphi})| \tag{3.28}$$

with $k \in \{1, ..., l\}$ for $l \in \mathbb{N}$. Integrating (3.28) yields

$$\begin{aligned} \Big| \int_{\varphi_k}^{\varphi_{k+1}} |D_\varphi Y(re^{i\varphi})|\, d\varphi - \int_{\varphi_k}^{\varphi_{k+1}} |D_\varphi J_V(X(e^{i\varphi_k})) X(re^{i\varphi})|\, d\varphi \Big| \\ \leq \varepsilon \int_{\varphi_k}^{\varphi_{k+1}} |D_\varphi X(re^{i\varphi})|\, d\varphi \end{aligned} \tag{3.29}$$

for $1-\delta(\varepsilon) \leq r \leq 1$ and $k \in \{1, ..., l\}$. Therefrom we obtain for $1-\delta(\varepsilon) \leq r \leq 1$

$$\begin{aligned}\Big| \int_{-\Theta}^{\Theta} |D_\varphi Y(re^{i\varphi})| d\varphi - \sum_{k=0}^{l} \int_{\varphi_k}^{\varphi_{k+1}} |D_\varphi J_V(X(e^{i\varphi_k}))X(re^{i\varphi})|\, d\varphi\Big| & \\ \leq \varepsilon \int_{-\Theta}^{\Theta} |D_\varphi X(re^{i\varphi})|\, d\varphi & \\ \leq \varepsilon \sup_{0\leq r\leq 1} \int_{-\Theta}^{\Theta} |D_\varphi X(re^{i\varphi})|\, d\varphi & \\ =: \varepsilon M, &\end{aligned}$$

where $M < \infty$ by assumption. Thus

$$\begin{aligned}&\Big| \int_{-\Theta}^{\Theta} |D_\varphi Y(re^{i\varphi})|\, d\varphi - \int_{-\Theta}^{\Theta} |D_\varphi Y(e^{i\varphi})|\, d\varphi \Big| \\ &\leq \sum_{k=0}^{l} \Big| \int_{\varphi_k}^{\varphi_{k+1}} |D_\varphi J_V(X(e^{i\varphi_k}))X(re^{i\varphi})| - |D_\varphi J_V(X(e^{i\varphi_k}))X(e^{i\varphi})|\, d\varphi \Big| \\ &\quad + 2\varepsilon M.\end{aligned} \tag{3.30}$$

By (3.26) and by semi-continuity of the arc length we find for each $k \in \{1, ..., l\}$

$$\lim_{r\to 1} \int_{\varphi_k}^{\varphi_{k+1}} |D_\varphi J_V(X(re^{i\varphi_k}))X(e^{i\varphi})|\, d\varphi = \int_{\varphi_k}^{\varphi_{k+1}} |D_\varphi J_V(X(e^{i\varphi_k}))X(e^{i\varphi})|\, d\varphi.$$

Taking the limit $r \to 1$ in (3.30) yields the claim. □

We now state a theorem which extends the main result of the article on the boundary behaviour of minimal surfaces of Heinz and Tomi [11].

Theorem 3.1.8. *Let $X \in H^1(B;\mathbb{R}^3) \cap C^2(B;\mathbb{R}^3) \cap C^0(\bar{B};\mathbb{R}^3)$ satisfy* (3.17), (3.18) *with boundary conditions* (3.19),(3.20) *where $g_k \in C^{2,\alpha}, 0 < \alpha < 1,\ k \in \{1,2\}$. Moreover, assume X solves*

$$\Delta X = \boldsymbol{H} f(X, \nabla X) \qquad \textit{in } B, \tag{3.31}$$

where $\boldsymbol{H} = \boldsymbol{H}(u,v)$ is a matrix whose entries are real, measurable and bounded functions, and where f complies with (3.2) *for a monotonically increasing continuous function μ. Finally, let $\Theta \in \mathbb{R}$ be so that $0 < \Theta \leq \pi$, and consider the* *transformation V as in* (3.22), (3.23)*. Assume that for each $|\varphi| \leq \Theta$ the Jacobian matrices $J_V\big(X(e^{i\varphi})\big)$ satisfy* (3.26)*. Then for all δ with $0 < \delta < \Theta$*

i. there holds $X \in C^{1,\alpha}(\bar{S}_{0,\delta};\mathbb{R}^3)$

ii. if $\boldsymbol{H} \in C^{\alpha}(\bar{B}_1)$ then there holds $X \in C^{2,\alpha}(\bar{S}_{0,\delta};\mathbb{R}^3)$

iii. if $g_k \in C^3$ for $k \in \{1,2\}$ then $X \in \bigcap_{0<\beta<1} C^{1,\beta}(\bar{S}_{0,\delta};\mathbb{R}^3)$

Proof. We choose R with $0 < R < 1$ so that V given by (3.22), (3.23) is defined on $X(\bar{S}_{R,\Theta})$ and has non-vanishing Jacobian J_V. We define $Y : \bar{S}_{R,\Theta} \to \mathbb{R}^3$ as $Y(re^{i\varphi}) := VX(re^{i\varphi})$.

Claim 3.1.8.1. Y solves an inequality of the form

$$|\Delta Y| \leq \alpha |\nabla Y|^2 \tag{3.32}$$

for a constant $\alpha > 0$ in $S_{R,\Theta}$.

Proof of Claim 3.1.8.1. Indeed, we have with (3.31) and (3.19) for $k \in \{1,2\}$

$$\begin{aligned} \Delta Y_k &= \Delta X_k - \big(g_k''(X_3)\big(\nabla X_3\big)^2 + g_k'(X_3)\Delta X_3\big) \\ &= \sum_{l=1}^{3} \big(H_{k,l} - g_k'(X_3)H_{3,l}\big) f_l(X,\nabla X) - g_k''(X_3)\big(\nabla X_3\big)^2. \end{aligned} \tag{3.33}$$

Hence with (3.2) for $k \in \{1,2\}$

$$|\Delta Y_k| \leq C|f(X,\nabla X)| + C\big|\nabla X_3\big|^2 \leq C|\nabla X|^2. \tag{3.34}$$

On the other hand, we find with (3.17), (3.18)

$$|\nabla X_3|^2 \leq \sum_{k=1}^{2} |\nabla X_k|^2.$$

This implies by the non-negativity of the absolute value

$$|\nabla X_3| \leq \big(\sum_{k=1}^{2} |\nabla X_k|^2\big)^{\frac{1}{2}} \leq \sum_{k=1}^{2} |\nabla X_k|.$$

Thus from (3.19)

$$\sum_{k=1}^{2}|\nabla X_k| \le \sum_{k=1}^{2}|\nabla Y_k| + C_0|\nabla X_3| \le \sum_{k=1}^{2}|\nabla Y_k| + C_0\sum_{k=1}^{2}|\nabla X_k|,$$

for some positive constant $C_0 < 1$. We may absorb the second term of the right-hand side on the left hand side, yielding

$$\sum_{k=1}^{2}|\nabla X_k| \le C_1\sum_{k=1}^{2}|\nabla Y_k|,$$

for $C_1 = \frac{1}{1-C_0} > 0$, so that

$$\sum_{k=1}^{3}|\nabla X_k| \le C_1\sum_{k=1}^{2}|\nabla Y_k| + \sum_{k=1}^{2}|\nabla X_k| \le 2C_1\sum_{k=1}^{2}|\nabla Y_k|. \tag{3.35}$$

Combining (3.35) with (3.34) yields the claim. □

Now we can invoke Lemma 3.1.6 since all required assumptions are met. This yields for $R < r < 1$ the estimate

$$\begin{aligned}
&\int_{-\Theta}^{\Theta}|D_r Y_3(re^{i\varphi})|\,d\varphi \\
&\le C\frac{1}{r}\int_{-\Theta}^{\Theta}\Big(|D_\varphi X(re^{i\varphi})|^{\frac{3}{4}}\big(|D_\varphi Y(re^{i\varphi})| - |D_\varphi Y_3(re^{i\varphi})|\big)^{\frac{1}{4}} \\
&\quad + |D_\varphi X(re^{i\varphi})|\big(|Y_1(re^{i\varphi})| + |Y_2(re^{i\varphi})|\big)\Big)\,d\varphi \\
&\le C\Bigg(\Big(\int_{-\Theta}^{\Theta}|D_\varphi X(re^{i\varphi})|\,d\varphi\Big)^{\frac{3}{4}}\Big(\int_{-\Theta}^{\Theta}\big(|D_\varphi Y(re^{i\varphi})| - |D_\varphi Y_3(re^{i\varphi})|\big)\,d\varphi\Big)^{\frac{1}{4}} \\
&\quad + \int_{-\Theta}^{\Theta}|D_\varphi X(re^{i\varphi})|\big(|Y_1(re^{i\varphi})| + |Y_2(re^{i\varphi})|\big)\,d\varphi\Bigg),
\end{aligned} \tag{3.36}$$

where we used $R < r < 1$ and Hölder's inequality. The boundary condition (3.20) guarantees the finiteness of $\int_{-\Theta}^{\Theta}|D_\varphi X(e^{i\varphi})|\,d\varphi$; in particular by assumption (3.26) for all $J_V(X(e^{i\varphi}))$ we have

$$\sup_{R<r<1}\int_{-\Theta}^{\Theta}|D_\varphi X(re^{i\varphi})|\,d\varphi \le C < \infty. \tag{3.37}$$

Therefore we can invoke Lemma 3.1.7, yielding

$$\lim_{r\to 1}\int_{-\Theta}^{\Theta}|D_\varphi Y(re^{i\varphi})|\,d\varphi=\int_{-\Theta}^{\Theta}|D_\varphi Y(e^{i\varphi})|\,d\varphi=\int_{-\Theta}^{\Theta}|D_\varphi Y_3(e^{i\varphi})|\,d\varphi, \tag{3.38}$$

where we used the boundary conditions

$$Y_k(e^{i\varphi})=0\quad\text{ for } |\varphi|\le\Theta,\quad k\in\{1,2\}, \tag{3.39}$$

occurring due to the slick choice of V.

Combining (3.36) with (3.37), (3.38) and (3.39) gives

$$\begin{aligned}
&\limsup_{r\to 1}\Big(\int_{-\Theta}^{\Theta}|D_rY_3(re^{i\varphi})|\,d\varphi\Big)^4\\
&\le C\limsup_{r\to 1}\Bigg(\int_{-\Theta}^{\Theta}|D_\varphi X(re^{i\varphi})|\big(|Y_1(re^{i\varphi})|+|Y_2(re^{i\varphi})|\big)\,d\varphi\\
&\quad+\Big(\int_{-\Theta}^{\Theta}|D_\varphi X(re^{i\varphi})|\,d\varphi\Big)^{\frac34}\Big(\int_{-\Theta}^{\Theta}|D_\varphi Y(re^{i\varphi})|\,d\varphi-\int_{-\Theta}^{\Theta}|D_\varphi Y_3(re^{i\varphi})|\,d\varphi\Big)^{\frac14}\Bigg)^4\\
&=C\limsup_{r\to 1}\Big(\int_{-\Theta}^{\Theta}|D_\varphi X(re^{i\varphi})|\,d\varphi\Big)^3\Big(\int_{-\Theta}^{\Theta}\big(|D_\varphi Y(re^{i\varphi})|-|D_\varphi Y_3(re^{i\varphi})|\big)\,d\varphi\Big)\\
&\le C\Big(\int_{-\Theta}^{\Theta}|D_\varphi Y_3(e^{i\varphi})|\,d\varphi-\liminf_{r\to 1}\int_{-\Theta}^{\Theta}|D_\varphi Y_3(re^{i\varphi})|\,d\varphi\Big)\\
&\le 0,
\end{aligned}$$

implying

$$\lim_{r\to 1}\int_{-\Theta}^{\Theta}|D_rY_3(re^{i\varphi})|\,d\varphi=0. \tag{3.40}$$

We can finally reap the fruits of our earlier results. First note that $Y=VX\in H^1(S_{R,\Theta};\mathbb{R}^3)\cap C^2(S_{R,\Theta};\mathbb{R}^3)\cap C^0(\bar S_{R,\Theta};\mathbb{R}^3)$ satisfies the differential inequality (3.32) with boundary conditions (3.39) and (3.40). Theorem 3.1.5 then gives
$Y\in\bigcap_{0<\beta<1}C^{1,\beta}(\bar S_{\tilde R,\delta};\mathbb{R}^3)$ for all $\tilde R,\delta$ with $R<\tilde R<1$ and $0<\delta<\Theta$. Then by definition of V (3.22), (3.23) we conclude *i.* and *iii.* in Theorem 3.1.8.

For *ii.* in Theorem 3.1.8 we introduce $Z_\varepsilon:=\{w=(u,v)\mid |w-e^{i\varphi_0}|<\varepsilon,|w|<1\}$ for $|\varphi_0|<\Theta$ and arbitrary $\varepsilon>0$. We assume now $\mathbf{H}\in C^\alpha(\bar B_1)$. Statement *i.* in

Theorem 3.1.8 gives

$$X_k \in C^{1,\alpha}(\bar{Z}_{\varepsilon_0};\mathbb{R}^3) \tag{3.41}$$

for $k \in \{1,2,3\},\ 0 < \varepsilon_0 < \varepsilon$.

First consider $k \in \{1,2\}$. By (3.33) we find $\Delta Y_k \in C^\alpha(\bar{Z}_{\varepsilon_0})$ due to (3.41). Moreover, (3.39) implies $Y_k(w) = 0$ for $w \in \bar{Z}_{\varepsilon_0}$ with $|w| = 1$. Thus estimates from Schauder theory, see Chapter 6 in [6], yield $Y_k \in C^{2,\alpha}(\bar{Z}_{\varepsilon_1})$ for some $0 < \varepsilon_1 < \varepsilon_0$. Since $g_k \in C^{2,\alpha}$ we obtain that $X_k = Y_k + g_k(X_3) \in C^{2,\alpha}(\bar{Z}_{\varepsilon_1};\mathbb{R}^3)$ for $k \in \{1,2\}$.

Now consider $k = 3$. Equation (3.40) implies $D_r Y_3(e^{i\varphi}) = 0$ for $|\varphi| < \Theta$. Thus we infer from (3.23) that $D_r X_3(e^{i\varphi})$ is of class $C^{1,\alpha}$ for $|\varphi| < \Theta$ since for $k \in \{1,2\}$ we have shown $X_k \in C^{2,\alpha}$ and we have assumed $g_k \in C^{2,\alpha}$. So we obtain $X_3(e^{i\varphi})$ is of class C^α. Furthermore, we have $\Delta X_3 \in C^\alpha(\bar{Z}_{\varepsilon_0};\mathbb{R}^3)$ since $\mathbf{H} \in C^\alpha(\bar{B}_1)$. With Schauder theory [6] we conclude $X_3 \in C^{2,\alpha}(\bar{Z}_{\varepsilon_1})$ for $0 < \varepsilon_1 < \varepsilon_0$.

Altogether we conclude $X \in C^{2,\alpha}(\bar{Z}_{\varepsilon_1};\mathbb{R}^3)$, yielding statement *ii.* in Theorem 3.1.8. □

3.2 Boundary Regularity for Minimal Surfaces

We can now investigate the boundary behaviour of minimal surfaces. With the methods established so far we state

Theorem 3.2.1. *Let $\Gamma \subset \mathbb{R}^3$ be a Jordan curve of class $C^{2,\alpha}$ for some $0 < \alpha < 1$, see Definition 3.0.2. Let $X : \bar{B} \to \mathbb{R}^3$ be a minimal surface of class $C^2(B;\mathbb{R}^3) \cap C^0(\bar{B};\mathbb{R}^3)$ bounded by Γ, that is $X = X(u,v) \in C^2(B;\mathbb{R}^3) \cap C^0(\bar{B};\mathbb{R}^3)$ satisfies* (2.7)–(2.9)*. Then $X \in C^{2,\alpha}(\bar{B};\mathbb{R}^3)$.*

Proof. First note that the isoperimetric inequality for minimal surfaces (2.30) yields the uniform boundedness of the Dirichlet integral. Thus with (2.8) we find that $X \in H^1(B;\mathbb{R}^3) \cap C^2(B;\mathbb{R}^3) \cap C^0(\bar{B};\mathbb{R}^3)$ satisfies (3.17), (3.18). Moreover (2.7) suggests to choose $\mathbf{H} = 0$ and f as the zero function on B in (3.31). With this choice we see that the assumptions on $\mathbf{H}$ and f in Theorem 3.1.8 are fulfilled.

Furthermore Γ is assumed to be of class $C^{2,\alpha}$, thus it is rectifiable. Hence we may approximate the curve with inscribed polygons. Since we can choose the approximation sufficiently fine, and since X maps the boundary of B onto Γ, the components of $X(e^{i\varphi})$ are given by functions $g_k \in C^{2,\alpha}, k \in \{1,2\}$ so that (3.19), (3.20) hold. Hence we can find a transformation V as in (3.22), (3.23), with the additional property that its Jacobian J_V is constant. We can then invoke Theorem III.3.4 in [3], which uses the harmonicity of X and the semi-continuity of the arc length, to find

$$\limsup_{r\to 1} \int_{\varphi_1}^{\varphi_2} |D_\varphi J_V X(re^{i\varphi})|\, d\varphi = \int_{\varphi_1}^{\varphi_2} |D_\varphi J_V X(e^{i\varphi})|\, d\varphi$$

for any $\varphi_1, \varphi_2 \in [-\pi, \pi)$ with $\varphi_1 < \varphi_2$.

Therefore we can use Theorem 3.1.8 to conclude the proof. $\square$

3.3 Boundary Regularity for H-Surfaces

In a final subsection we extend the considerations of Heinz and Tomi in [11] to conclude with a statement on the boundary behaviour of H-surfaces.

Theorem 3.3.1. *Let $\Gamma \subset B \subset \mathbb{R}^3$ be a Jordan curve of class $C^{2,\alpha}$ for $0 < \alpha < 1$. Let $H \in \mathbb{R}$ be such that $|H| < \frac{1}{2}$. Let $X : \bar{B} \to \mathbb{R}^3$ be a surface of constant mean curvature H of class $C^2(B;\mathbb{R}^3) \cap C^0(\bar{B};\mathbb{R}^3)$ bounded by Γ; that is $X = X(u,v) \in C^2(B;\mathbb{R}^3) \cap C^0(\bar{B};\mathbb{R}^3)$ satisfies* (2.31)–(2.33). *Then $X \in C^{2,\alpha}(\bar{B};\mathbb{R}^3)$.*

Proof. Note that we cannot apply the isoperimetric inequality for minimal surfaces to obtain a uniform a-priori bound for Dirichlet's integral. However, there exists a variant for H-surfaces due to Heinz [7]. Indeed, there holds

$$\frac{1}{2}\int_B |\nabla X|^2\, dw \le \frac{1}{4\pi}\frac{1+|H|}{1-|H|} L(\Gamma)^2 < \infty,$$

where we refer to Theorem 3 in [7]. We infer with (2.32) that $X \in H^1(B;\mathbb{R}^3) \cap C^2(B;\mathbb{R}^3) \cap C^0(\bar{B};\mathbb{R}^3)$ satisfies (3.17), (3.18). Moreover, we choose $\mathbf{H}$ in (3.31) as the identity matrix in $\mathbb{R}^{3,3}$, and we set $f(X, \nabla X) := 2HX_u \wedge X_v$. Then by (2.5) we have that f satisfies (3.2) with $\mu(\|X\|_{L^\infty}) = H$. Furthermore, since X maps the boundary of B onto Γ, the components of $X(e^{i\varphi})$ are given by functions $g_k \in$

$C^{2,\alpha}, k \in \{1,2\}$ so that (3.19), (3.20) hold. Hence we can find a transformation V as in (3.22), (3.23). Finally since $|H| < \frac{1}{2}$ we obtain the relation

$$\limsup_{r \to 1} \int_{\varphi_1}^{\varphi_2} |D_\varphi J_V X(re^{i\varphi})| \, d\varphi = \int_{\varphi_1}^{\varphi_2} |D_\varphi J_V X(e^{i\varphi})| \, d\varphi$$

for any $\varphi_1, \varphi_2 \in [-\pi, \pi)$ with $\varphi_1 < \varphi_2$, as demonstrated in Theorem 1 in [7].

Therefore we can invoke Theorem 3.1.8 to conclude the desired regularity. $\square$

As a concluding remark I want to emphasise that the methods used in this chapter are clearly not the most straight-forward way to reach the conclusion, nor do they deliver the most general result, confer Theorem 3.0.1. Indeed the reasoning can be simplified using more modern tools, as Struwe has demonstrated throughout his research on minimal surfaces and elliptic regularity theory [31, 30, 29]. Using differential geometrical considerations, we can transform minimal surfaces and H-surfaces into an elliptic system with quadratic growth in the gradient (Theorem I.5.1 and Theorem III.5.5 in [31]). Then the $C^{2,\alpha}$-regularity can be obtained by Theorem 2.8 in [29]. Even though all these results lead in a simple and beautiful way to the same conclusions, this thesis attempts to discuss Heinz and Tomi's approach as I understand it.

Bibliography

[1] Ahlfors, Lars Valerian: Complex Analysis. McGraw-Hill, Inc. (1979).

[2] Carmo, Manfredo Pedigao do: Differential Geometry of Curves and Surfaces. Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1976).

[3] Courant, Richard: Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces. Dover Publications, INC. Mineola, New York (2005).

[4] Douglas, Jesse: Solution of the problem of Plateau. Trans. Am. Math. Soc. 33, 263-321 (1931).

[5] Garnett, John B.; Marshall, Donald E.: Harmonic Measure. Cambridge University Press (2005).

[6] Gilbarg, David; Trudinger, Neil S.: Elliptic Partial Differential Equations of Second Order, Springer Verlag, Berlin–Heidelberg–New York (2001).

[7] Heinz, Erhard: An Inequality of Isoperimetric Type for Surfaces of Constant Mean Curvature. Arch. Rat. Mech. Anal. 33, 155–168 (1969).

[8] Heinz, Erhard: On the nonexistence of a surface of constant mean curvature with finite area and prescribed rectifiable boundary. Arch. Rat. Mech. Anal. 35, 249–252 (1969).

[9] Heinz, Erhard: Über das Randverhalten quasilinearer elliptischer Systeme mit isothermen Parametern. Math. Zeitschrift, 113, 99–105 (1970).

[10] Heinz, Erhard: Über die Existenz einer Fläche konstanter mittlerer Krümmung bei vorgegebener Berandung. Math. Annalen, Bd. 127, 258–287 (1954).

[11] Heinz, Erhard; Tomi, Friedrich: Zu einem Satz von Hildebrandt über das Randverhalten von Minimalflächen. Mathematische Zeitschrift, Vol. 111, 372–386 (1969).

[12] Hildebrandt, Stefan: Boundary Behaviour of Minimal Surfaces. Arch. Rational Mech. Anal. 35, 47–82 (1969).

[13] Hildebrandt, Stefan: Boundary Value Problems for Minimal Surfaces. Chapter III. in Ossermann, Robert: Geometry V: Minimal Surfaces. Encyclopaedia of Mathematical Sciences, Volume 90, Springer (1997).

[14] Hildebrandt, Stefan: On the Plateau problem for surfaces of constant mean curvature. Comm. Pure Applied Mathematics, 23, 97–114 (1970).

[15] Hildebrandt, Stefan: Über das Randverhalten von Minimalflächen. Math. Annalen 165, 1–18 (1966).

[16] Kellogg, O. D.: Harmonic functions and Green's integral. Transact. Amer. Math. Soc. 13, 109–132 (1912).

[17] Lewy, Hans: On the boundary behavior of minimal surfaces. Proc. Nat. Acad. Sci. USA, 37, 103–110 (1951).

[18] Maxwell, Clerk J.: Plateau on Soap Bubbles, Nature, Vol. x., No. 242 (1874).

[19] Morrey, Charles Bradfield: Multiple integrals in the calculus of variations. Springer Verlag, Berlin-Heidelberg-New York (1966).

[20] Morrey, Charles Bradfield: The problem of Plateau on a Riemannian manifold. Ann. Math. 49, 807–851 (1948).

[21] Nitsche, Johannes C. C.: The Boundary Behavior of Minimal Surfaces. Kellogg's Theorem and Branch Points on the Boundary. Inventiones Math. 8, 313–333 (1969).

[22] Nitsche, Johannes C. C.: Vorlesungen über Minimalflächen. Springer Verlag, Berlin-Heidelberg-New York (1975).

[23] Radò, Tibor: The isoperimetric inequality and the Lebesgue definition of the surface area. Trans. AMS 61, 530–555 (1947).

[24] Radò, Tibor: The problem of least area and the problem of Plateau. Mathematische Zeitschrift 32, 763–795 (1930).

[25] Rudin, Walter: Principles of Mathematical Analysis (3rd ed.). New York: McGraw-Hill, 113 (1976).

[26] Schoen, Richard; Yau Shing-Tung: Lectures on harmonic maps, Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA (1997).

[27] Simon, Leon: The Minimal Surface Equation. Chapter IV. in Ossermann, Robert: Geometry V: Minimal Surfaces. Encyclopaedia of Mathematical Sciences, Vol. 90, Springer (1997).

[28] Struwe, Michael: Analysis III: Mass und Integral. Lecture Notes, ETH Zurich (2013).

[29] Struwe, Michael: Elliptic Regularity Theory. Lecture Notes, ETH Zurich (2020).

[30] Struwe, Michael: Funktionalanalysis I und II. Lecture Notes, ETH Zurich (2019/2020).

[31] Struwe, Michael: Plateau's Problem and the Calculus of Variations. Princeton University Press (1989).

[32] Tomi, Friedrich: Ein einfacher Beweis eines Regularitätssatzes für schwache Lösungen gewisser elliptischer Systeme. Math. Zeitschrift, 112, 214–218 (1969).

[33] Vekua, Ilia: Verallgemeinerte analytische Funktionen. Berlin: Akademie-Verlag (1963).

[34] Wente, Henry C.: An Existence Theorem for Surfaces of Constant Mean Curvature. Journal of Mathematical Analysis and Applications, 26, 318–344 (1969).

[35] Weyl, Hermann; The method of orthogonal projections in potential theory. Duke Mathematical Journal, 7, 411–444 (1940).